

ELECTROMAGNETIC THEORY

for

CSIR-UGC-NET/JRF | GATE

TIFR | JEST | BARC | IISc | JNU | SLET | M.Sc. Entrance | Ph.D Entrance

First Edition : July 2015

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Chapter 1

Mathematics Involved in Electromagnetic Theory

Gradient of a scalar function:

If $\phi(x, y, z)$ be a scalar function then $\hat{i} \frac{\partial \phi}{\partial x} + \hat{j} \frac{\partial \phi}{\partial y} + \hat{k} \frac{\partial \phi}{\partial z}$ is called the gradient of the scalar function ϕ .

And is denoted by $\text{grad } \phi$

Thus,

$$\text{grad } \phi = \hat{i} \frac{\partial \phi}{\partial x} + \hat{j} \frac{\partial \phi}{\partial y} + \hat{k} \frac{\partial \phi}{\partial z}$$

$$\text{grad } \phi = \left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) \phi(x, y, z)$$

$$\text{grad } \phi = \nabla \phi$$

Geometrical Meaning of Gradient, Normal:

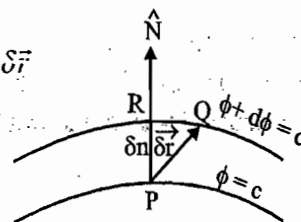
If a surface $\phi(x, y, z) = c$ passes through a point P. The value of the function at each point on the surface is the same as at P. Then such a surface is called a level surface through P. For example, if $\phi(x, y, z)$ represents potential at the point P, then equipotential surface $\phi(x, y, z) = c$ is a level surface.

Two level surfaces can not intersect.

Let the level surface pass through the point P at which the value of the function is ϕ . Consider another level surface passing through Q, where the value of the function is $\phi + d\phi$.

Let $\vec{r}$ and $\vec{r} + \delta\vec{r}$ be the position vector of P and Q then $\overline{PQ} = \delta\vec{r}$

$$\begin{aligned} \vec{\nabla} \phi \cdot d\vec{r} &= \left(\hat{i} \frac{\partial \phi}{\partial x} + \hat{j} \frac{\partial \phi}{\partial y} + \hat{k} \frac{\partial \phi}{\partial z} \right) \cdot (\hat{i} dx + \hat{j} dy + \hat{k} dz) \\ &= \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy + \frac{\partial \phi}{\partial z} dz = d\phi \end{aligned}$$



... (1)

If Q lies on the level surface of P, then $d\phi = 0$

Equation (1) becomes, $\vec{\nabla} \phi \cdot d\vec{r} = 0$, then $\vec{\nabla} \phi$ is perpendicular to $d\vec{r}$ (tangent)

Hence, $\vec{\nabla} \phi$ is normal to the surface $\phi(x, y, z) = c$

The Divergence:

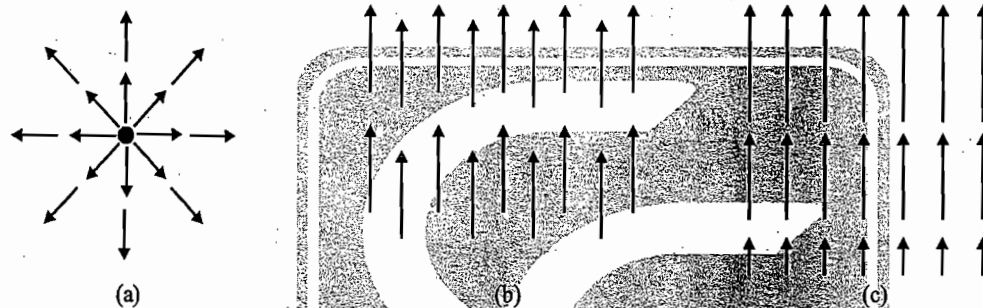


$$\begin{aligned}\vec{\nabla} \cdot \vec{v} &= \left(\hat{x} \frac{\partial}{\partial x} + \hat{y} \frac{\partial}{\partial y} + \hat{z} \frac{\partial}{\partial z} \right) \cdot (v_x \hat{x} + v_y \hat{y} + v_z \hat{z}) \\ &= \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\end{aligned}$$

Observe that the divergence of a vector function $\vec{v}$ is itself a scalar $\vec{\nabla} \cdot \vec{v}$.

Geometrical interpretation :

The name divergence is well chosen, for $\vec{\nabla} \cdot \vec{v}$ is a measure of how much the vector $\vec{v}$ spreads out (diverges) from the point in question. For example, the vector function in figure (a) has a large (positive) divergence (if the arrows pointed in, it would be a large negative divergence), the function in figure (b) has zero divergence, and the function in figure (c) again has a positive divergence.



The Curl:

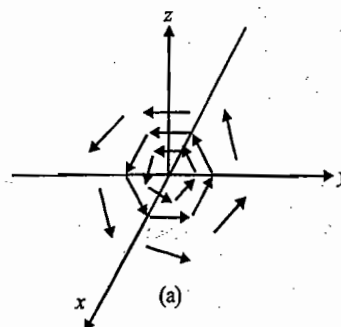
From the definition ∇ we construct the curl:

$$\vec{\nabla} \times \vec{v} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_x & v_y & v_z \end{vmatrix} = \hat{x} \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z} \right) + \hat{y} \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x} \right) + \hat{z} \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \right)$$

Notice that the curl of a vector function $\vec{v}$ is, like any cross product, a vector. Curl of a scalar does not exist.

Geometrical interpretation :

The name curl is also well chosen, for $\vec{\nabla} \times \vec{v}$ is a measure of how much the vector $\vec{v}$ 'curls around' the point in question. Thus the three functions in figure (a) have a substantial curl, pointing in the z-direction, as the natural right-hand rule would suggest.



**Product rules:**

- (i) $\vec{\nabla}(fg) = f \vec{\nabla} g + g \vec{\nabla} f$
- (ii) $\vec{\nabla}(\vec{A} \cdot \vec{B}) = \vec{A} \times (\vec{\nabla} \times \vec{B}) + \vec{B} \times (\vec{\nabla} \times \vec{A}) + (\vec{A} \cdot \vec{\nabla}) \vec{B} + (\vec{B} \cdot \vec{\nabla}) \vec{A}$,
- (iii) $\vec{\nabla} \cdot (f \vec{A}) = f (\vec{\nabla} \cdot \vec{A}) + \vec{A} \cdot (\vec{\nabla} f)$
- (iv) $\vec{\nabla} \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot (\vec{\nabla} \times \vec{A}) - \vec{A} \cdot (\vec{\nabla} \times \vec{B})$,
- (v) $\vec{\nabla} \times (f \vec{A}) = f (\vec{\nabla} \times \vec{A}) - \vec{A} \times (\vec{\nabla} f)$
- (vi) $\vec{\nabla} \times (\vec{A} \times \vec{B}) = (\vec{B} \cdot \vec{\nabla}) \vec{A} - (\vec{A} \cdot \vec{\nabla}) \vec{B} + \vec{A} (\vec{\nabla} \cdot \vec{B}) - \vec{B} (\vec{\nabla} \cdot \vec{A})$

Problem-1: Find the gradients of the following functions:

- (a) $f(x, y, z) = x^2 + y^3 + z^4$ (b) $f(x, y, z) = x^2 y^3 z^4$ (c) $f(x, y, z) = e^x \sin(y) \ln(z)$

Problem-2: Let ' $\vec{r}$ ' be the separation vector from a fixed point (x', y', z') to the point (x, y, z) and let ' r ' be its length. Show that

- (a) $\vec{\nabla}(r^2) = 2\vec{r}$ (b) $\vec{\nabla}(1/r) = -\vec{r}/r^3$ (c) what is the general formula $\vec{\nabla}(r^n)$?

Problem-3: Calculate the divergence of the following vector functions

- (a) $\vec{v}_a = x^2 \hat{x} + 3xz^2 \hat{y} - 2xz \hat{z}$ (b) $\vec{v}_b = xy \hat{x} + 2yz \hat{y} + 3zx \hat{z}$
- (c) $\vec{v}_c = y^2 \hat{x} + (2xy + z^2) \hat{y} + 2yz \hat{z}$

Problem-4: The vector function, $\vec{v} = \frac{\vec{r}}{r^2}$ compute its divergence.

The gradient, the divergence, and the curl are the only first derivatives we can make with $\vec{\nabla}$; by applying $\vec{\nabla}$ twice we can construct five species of second derivatives. The gradient $\vec{\nabla} T$ is a vector, so we can take the divergence and curl of it.

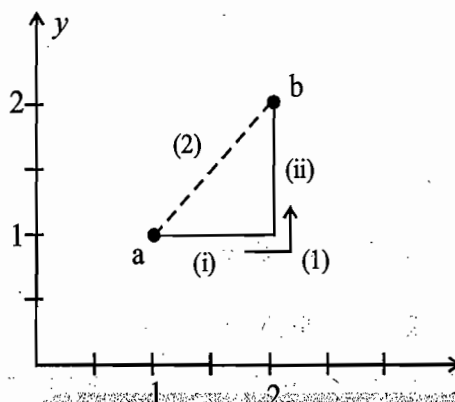
- (1) Divergence of gradient: $\vec{\nabla} \cdot (\vec{\nabla} T) = \nabla^2 T$ (Laplacian of T)
- (2) Curl of gradient: $\vec{\nabla} \times (\vec{\nabla} T) = 0$
- (3) Gradient of divergence: $\vec{\nabla}(\vec{\nabla} \cdot \vec{v})$ (This is not same as laplacian)
- (4) Divergence of curl: $\vec{\nabla} \cdot (\vec{\nabla} \times \vec{v}) = 0$
- (5) Curl of curl: $\vec{\nabla} \times (\vec{\nabla} \times \vec{v}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{v}) - \nabla^2 \vec{v}$

Problem-5: Calculate the Laplacian of the following functions:

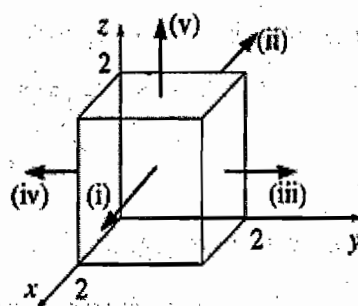
- (a) $T_a = x^2 + 2xy + 3z + 4$ (b) $T_b = \sin x \sin y \sin z$
- (c) $T_c = e^{-5x} \sin 4y \cos 3z$ (d) $v = x^2 \hat{x} + 3xz^2 \hat{y} - 2xz \hat{z}$



Problem-6: Calculate the line integral of the function $\vec{v} = y^2\hat{x} + 2x(y+1)\hat{y}$ from the point $a = (1, 1, 0)$ to the point $b = (2, 2, 0)$, along the paths (1) and (2) in the following figure. What is $\oint \vec{v} \cdot d\vec{l}$ for the loop that goes from 'a' to 'b' along (1) and returns to 'a' along (2)?



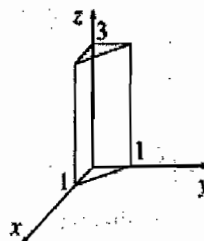
Problem-7: Calculate the surface integral of $\vec{v} = 2xz\hat{x} + (x+2)\hat{y} + y(z^2-3)\hat{z}$ over five sides (excluding the bottom) of the cubical box (side 2) in figure. Let 'upward and outward' be the positive direction, as indicated by the arrows.



Problem-8: Calculate the volume integral of $T = xyz^2$ over the prism in figure.

Soln. You can do the three integrals in any order. Let's do 'x' first: it runs from 0 to $(1-y)$; then y(it goes from 0 to 1); and finally z (0 to 3):

$$\int T d\tau = \int_0^3 z^2 \left\{ \int_0^1 y \left[\int_0^{1-y} x dx \right] dy \right\} dz = \frac{1}{2} \int_0^3 z^2 dz \int_0^1 (1-y^2)^2 y dy = \frac{1}{2} \times 9 \times \frac{1}{12} = \frac{3}{8}$$



Problem-9: Calculate the line integral of the function $\vec{v} = x^2\hat{x} + 2yz\hat{y} + y^2\hat{z}$ from the origin to the point $(1, 1, 1)$ by three different routes:

(a) $(0, 0, 0) \rightarrow (1, 0, 0) \rightarrow (1, 1, 0) \rightarrow (1, 1, 1)$

(b) $(0, 0, 0) \rightarrow (0, 0, 1) \rightarrow (0, 1, 1) \rightarrow (1, 1, 1)$

(c) The direct straight line.

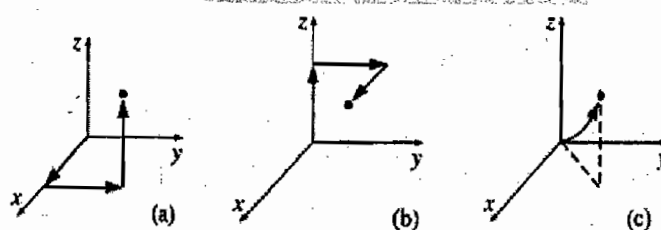
Example-10: Let $T = xy^2$, and take point 'a' to be the origin $(0, 0, 0)$ and 'b' the point $(2, 1, 0)$. Check the fundamental theorem for gradients.

Problem-11: Check the fundamental theorem for the gradients, using $T = x^2 + 4xy + 2yz^3$, the points $a = (0, 0, 0)$, $b = (1, 1, 1)$ and the three paths in figures (a), (b) and (c) are

(a) $(0, 0, 0) \rightarrow (1, 0, 0) \rightarrow (1, 1, 0) \rightarrow (1, 1, 1)$

(b) $(0, 0, 0) \rightarrow (0, 0, 1) \rightarrow (0, 1, 1) \rightarrow (1, 1, 1)$

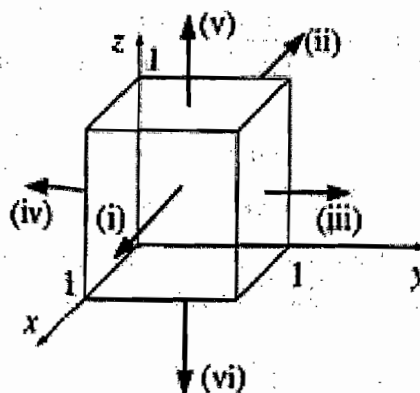
(c) The parabolic path $z = x^2$; $y = x$



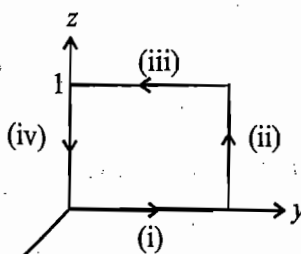
Problem-12: Check the divergence theorem using the function

$$\vec{v} = y^2 \hat{x} + (2xy + z^2) \hat{y} + (2yz) \hat{z}$$

and the unit cube situated at the origin has shown in the following figure.

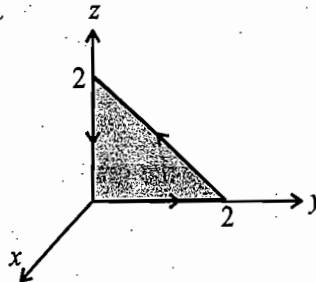


Problem-13: Suppose $\vec{v} = (2xz + 3y^2) \hat{y} + (4yz^2) \hat{z}$. Check Stokes' theorem for the square surface shown in figure.





Problem-14: Test Stokes' theorem for the function $\vec{v} = (xy)\hat{x} + (2yz)\hat{y} + (3zx)\hat{z}$, using the triangular shaded area of figure.



Problem-15: Evaluate the integral

$$\int_0^{\infty} x e^{-x} dx$$

Formulas for gradient divergence and curl for spherical coordinates.

Gradient: $\vec{\nabla}T = \frac{\partial T}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial T}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial T}{\partial \phi}\hat{\phi}$

Divergence: $\vec{\nabla} \cdot \vec{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi}$

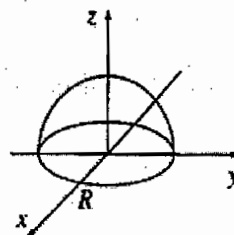
Curl: $\vec{\nabla} \times \vec{v} = \frac{1}{r\sin\theta} \left[\frac{\partial}{\partial \theta}(\sin\theta v_\phi) - \frac{\partial v_\theta}{\partial \phi} \right] \hat{r} + \frac{1}{r} \left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{\partial}{\partial r}(r v_\phi) \right] \hat{\theta} + \frac{1}{r} \left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial \theta} \right] \hat{\phi}$

Laplacian: $\nabla^2 T = \frac{1}{r^2}\frac{\partial}{\partial r} \left(r^2 \frac{\partial T}{\partial r} \right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta} \left(\sin\theta \frac{\partial T}{\partial \theta} \right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 T}{\partial \phi^2}$

Problem-16: Compute the divergence of the function

$$\vec{v} = (r \cos \theta)\hat{r} + (r \sin \theta)\hat{\theta} + (r \sin \theta \cos \phi)\hat{\phi}$$

Check the divergence theorem for this function, using as your volume the inverted hemispherical bowl of radius R , resting on the xy -plane and centered at the origin (Figure)

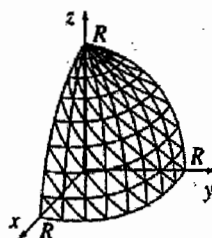


Problem-17: Find the formulas for r, θ, ϕ in terms of $\hat{x}, \hat{y}, \hat{z}$.

Problem-19: Express the cylindrical unit vectors $\hat{s}, \hat{\phi}, \hat{z}$ in terms of $\hat{x}, \hat{y}, \hat{z}$. Invert your formulas to get $\hat{x}, \hat{y}, \hat{z}$ in terms of $\hat{s}, \hat{\phi}, \hat{z}$.

Problem-20: Check the divergence theorem for the function

$$\vec{v} = r^2 \cos \theta \hat{r} + r^2 \cos \phi \hat{\theta} - r^2 \cos \theta \sin \phi \hat{\phi}$$

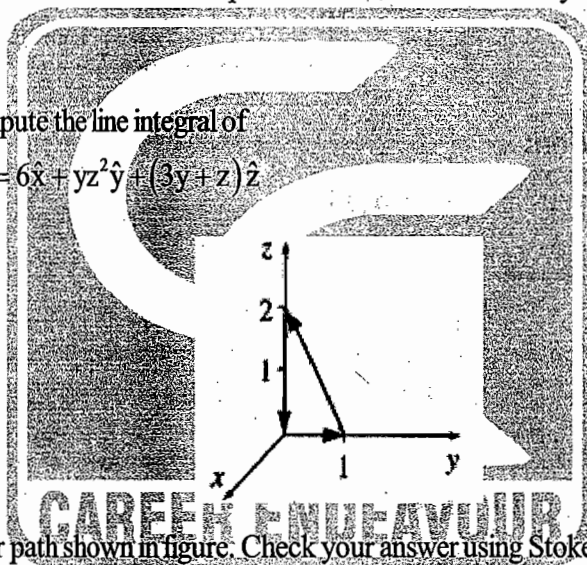


using as your volume one octant of the sphere of radius R . Make sure you include the entire surface.

Ans. $\pi R^4/4$.

Problem-21: Compute the line integral of

$$\vec{v} = 6x\hat{x} + yz^2\hat{y} + (3y + z)\hat{z}$$



along the triangular path shown in figure. Check your answer using Stoke's theorem.

Ans. $8/3$.

Coordinate System:

Coordinate system	$\hat{u}_1$	$\hat{u}_2$	$\hat{u}_3$	h_1	h_2	h_3	ds	dv
Cartesian	$\hat{x}$	$\hat{y}$	$\hat{z}$	1	1	1	$dx dy$ or $dy dz$ or $dz dx$	$dx dy dz$
Spherical	$\hat{r}$	$\hat{\theta}$	$\hat{\phi}$	1	r	$r \sin \theta$	$rd\theta r \sin \theta d\phi$	$r^2 \sin \theta dr d\theta d\phi$
Cylindrical	$\hat{\rho}$	$\hat{\theta}$	$\hat{z}$	1	ρ	1	$\rho d\theta dz$	$\rho d\rho d\theta dz$

$$(i) \nabla = \frac{1}{h_1} \frac{\partial}{\partial u_1} \hat{e}_1 + \frac{1}{h_2} \frac{\partial}{\partial u_2} \hat{e}_2 + \frac{1}{h_3} \frac{\partial}{\partial u_3} \hat{e}_3$$

$$(ii) \vec{\nabla} \cdot \vec{A} = \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial u_1} (h_2 h_3 A_1) + \frac{\partial}{\partial u_2} (h_1 h_3 A_2) + \frac{\partial}{\partial u_3} (h_1 h_2 A_3) \right]$$



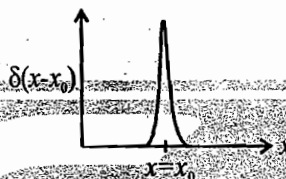
$$(iii) \vec{\nabla} \times \vec{A} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} h_1 \hat{e}_1 & h_2 \hat{e}_2 & h_3 \hat{e}_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ h_1 A_1 & h_2 A_2 & h_3 A_3 \end{vmatrix}$$

$$(iv) \vec{\nabla}^2 \phi = \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial u_1} \left(\frac{h_2 h_3}{h_1} \frac{\partial \phi}{\partial u_1} \right) + \frac{\partial}{\partial u_2} \left(\frac{h_1 h_3}{h_2} \frac{\partial \phi}{\partial u_2} \right) + \frac{\partial}{\partial u_3} \left(\frac{h_1 h_2}{h_3} \frac{\partial \phi}{\partial u_3} \right) \right]$$

Dirac-delta function:

(A) One dimensional delta function:

$$\delta(x-x_0) = \begin{cases} 0 & \text{for } x \neq x_0 \\ \infty & \text{for } x = x_0 \end{cases}$$



Properties of delta function:

(1) The delta function is even $\delta(-x) = \delta(x)$

(2) $\delta(ax) = \frac{1}{|a|} \delta(x)$ ($a \neq 0$)

(3) $\int_{-\infty}^{\infty} \delta(x) dx = 1$

(4) $\int_a^b f(x) \delta(x-x_0) dx = \begin{cases} f(x_0) & \text{if } a < x_0 < b \\ 0 & \text{elsewhere} \end{cases}$

(5) $\int \delta(a-x) \delta(x-b) dx = \delta(a-b)$

(6) $\delta[(x-a)(x-b)] = \frac{1}{|a-b|} [\delta(x-a) + \delta(x-b)]$ ($a \neq b$)

(7) $\delta(x^2 - a^2) = \frac{1}{2|a|} [\delta(x-a) + \delta(x+a)]$ ($a \neq 0$)

(8) $\int_a^b f(x) \delta^n(x-x_0) dx = (-1)^n f^n(x_0)$ $a < x_0 < b$

(B) Three dimensional delta function:

(1) The three dimensional form delta function in cartesian coordinates is

$$\delta(\vec{r} - \vec{r}_0) = \delta(x-x_0) \delta(y-y_0) \delta(z-z_0)$$

(2) In spherical coordinate

$$\delta(\vec{r} - \vec{r}_0) = \frac{1}{r^2 \sin \theta} \delta(r-r_0) \delta(\theta-\theta_0) \delta(\phi-\phi_0)$$

$$(3) \int_V \delta(\vec{r} - \vec{r}_0) dV = \begin{cases} 1 & \text{if } r_0 \text{ is in } V \\ 0 & \text{if } r_0 \text{ is not in } V \end{cases}$$

$$(4) \int_V f(\vec{r}) \delta(\vec{r} - \vec{r}_0) dV = \begin{cases} f(r_0) & \text{if } r_0 \text{ is in } V \\ 0 & \text{if } r_0 \text{ is not in } V \end{cases}$$

$$\int_V \vec{\nabla} \cdot \frac{\vec{r}}{r^3} dV = \int_S \frac{\vec{r}}{r^3} \cdot d\vec{s} = \frac{1}{r^2} \int ds = \frac{4\pi r^2}{r^2} = 4\pi$$

$$\vec{\nabla} \cdot \frac{\vec{r}}{r^3} = \vec{\nabla} \cdot \vec{\nabla} \left(-\frac{1}{r} \right) = -\nabla^2 \frac{1}{r} = 4\pi \delta(\vec{r}) \Rightarrow \delta(\vec{r}) = -\frac{1}{4\pi} \nabla^2 \left(\frac{1}{r} \right)$$



Chapter 2

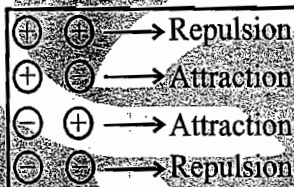
Electric field & Potential in vacuum

Charge

Charge is one of the fundamental properties of elementary particles which are the basic building blocks of universe.

we know masses attract each other from law of gravitation. Similarly charged particles also attract or repulse each other depending on the nature of charges.

There are two kinds of charges. Positive (+) and negative (-). Similar kinds of charges repel each other where as the opposite kinds attract. Total charge in the universe is conserved. Any charge is the multiple of basic unit charge is 1.6×10^{-19} Coulomb.



How to calculate total charge:

Case-I : Linear charge distribution, $dq = \lambda dl$ or $q = \int \lambda dl$

Case-II : Surface charge distribution, $dq = \sigma ds$ or $q = \iint \sigma ds$

Case-III: Volume charge distribution, $dq = \rho dv$ or $q = \iiint \rho dv$

Density of a point charge is $q\delta(r)$

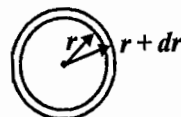
Example: Suppose the electron cloud around the nucleus of charge $+e$ in an atom can be represented by the following spherical charge distributors.

$$\rho(r) = \frac{-e}{\pi r_0} \cdot e^{-2r/r_0}$$

where r_0 is called the Bohr radius. Calculate the total charge of the cloud.

Soln. We consider a thin spherical shell of radius r and thickness dr . Its volume $4\pi r^2 dr$ and charge within it is

$\rho(r) \cdot 4\pi r^2 dr$. Therefore, total charge of the cloud is



$$Q = \int_0^\infty \rho(r) 4\pi r^2 dr = \frac{-e}{\pi r_0} \cdot 4\pi \int_0^\infty r^2 e^{-2r/r_0} dr$$

Putting $\frac{2r}{r_0} = x$, we get $Q = \frac{-e}{\pi r_0} \int_0^\infty \frac{r_0^3}{4} x^2 e^{-x} dx = \frac{-e}{\pi r_0} \cdot \frac{r_0^3}{4} \int_0^\infty x^2 e^{-x} dx = \frac{-e}{\pi r_0} \cdot \frac{r_0^3}{4} \cdot 2 = \frac{-e}{2\pi}$



Coulomb's law:

The force between two stationary point charge is proportional to the product of charges, inversely proportional to the square of the distance between them and it acts along the line joining the charges.

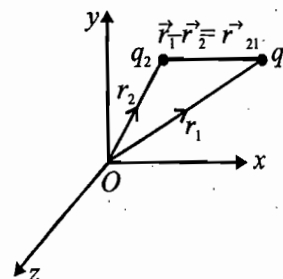
Let $\vec{r}_1$ and $\vec{r}_2$ are position of two charges q_1 and q_2 .

Therefore, according to Coulomb law the force on q_2 due to q_1 is

$$\vec{F}_{21} = \frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{r_{21}^2} \hat{r}_{21} \quad \dots (1)$$

The force on q_1 due to q_2 is

$$\vec{F}_{12} = \frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{r_{12}^2} \hat{r}_{12} = -\frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{r_{21}^2} \hat{r}_{21} \quad \dots (2)$$



where, ϵ_0 is dielectric permittivity of free space

$$\epsilon_0 = 8.85 \times 10^{-12} \text{ C}^2/\text{N-m}^2$$

From equation (1) and (2), we get

$$\vec{F}_{21} = -\vec{F}_{12}$$

Principle of Superposition:

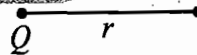
Let we have N point charges $q_0, q_1, q_2, q_3, \dots, q_N$. The total force on q_0 due to all other charges.

$$\vec{F}_0 = \vec{F}_{01} + \vec{F}_{02} + \vec{F}_{03} + \dots + \vec{F}_{0N} = \sum_{j=1}^N \vec{F}_{0j}$$

Electric field:

If we keep a charge in the space, the region within other charge feel force this region is called electric field. And the force that acts on the unit charge at a particular point is called electric field intensity ($\vec{E}$) at that point.

$$\vec{E} = \frac{1}{4\pi\epsilon_0} \frac{Q}{r^2} \hat{r} \quad \dots (3)$$



For continuous charge distribution :

$$(1) \text{ For linear charge distribution, } \vec{E} = \frac{1}{4\pi\epsilon_0} \int \frac{\lambda d\ell}{R^2} \hat{R}$$

$$(2) \text{ Surface charge distribution, } \vec{E} = \frac{1}{4\pi\epsilon_0} \int \frac{\sigma ds}{R^2} \hat{R}$$

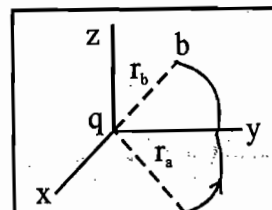
$$(3) \text{ Volume charge distribution, } \vec{E} = \frac{1}{4\pi\epsilon_0} \int \frac{\rho dv}{R^2} \hat{R}$$

Electric Potential:

Curl of $\vec{E}$

we know field of a point charge at the origin

$$\vec{E} = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \hat{r}$$





$$= \frac{1}{4\pi\epsilon_0} \int_a^b \frac{q}{r^2} dr = \frac{1}{4\pi\epsilon_0} \left(\frac{q}{r_a} - \frac{q}{r_b} \right)$$

If we take the integral along a closed path ($r_a = r_b$) then

$$\oint \vec{E} \cdot d\vec{l} = 0$$

Applying Stokes theorem $\oint \vec{E} \cdot d\vec{l} = \int_S (\vec{\nabla} \times \vec{E}) \cdot d\vec{S} = 0$

Hence, $\vec{\nabla} \times \vec{E} = 0$

This result can be generalised for any charge distribution using the principle of superposition. The general result is $\vec{\nabla} \times \vec{E} = 0$... (4)

Electric Potential:

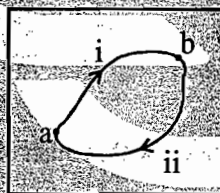
We have proved for any closed path the integral $\oint \vec{E} \cdot d\vec{l} = 0$. Take the closed path in figure above. Suppose

you go from a to b by path 1 and come back from b to a by path 2. The integral $\oint \vec{E} \cdot d\vec{l} = 0$

$$\int_{a \text{ (path i)}}^b \vec{E} \cdot d\vec{l} + \int_{b \text{ (path ii)}}^a \vec{E} \cdot d\vec{l} = 0$$

Or, $\int_{a \text{ path i}}^b \vec{E} \cdot d\vec{l} + \int_{b \text{ path ii}}^a \vec{E} \cdot d\vec{l} = 0$

Or, $\int_{a \text{ path i}}^b \vec{E} \cdot d\vec{l} = \int_{a \text{ path ii}}^b \vec{E} \cdot d\vec{l}$



(Fig. 1.22)

So, the line integral $\int_a^b \vec{E} \cdot d\vec{l}$ is same for any path joining (a) and (b).

So we define a function

$$V(\vec{r}) = - \int_{\infty}^{\vec{r}} \vec{E} \cdot d\vec{l} \quad \dots (5)$$

There O is a reference point which is generally taken to be at infinity.

The potential difference between 2 points

$$\begin{aligned} V(\vec{b}) - V(\vec{a}) &= - \int_{\infty}^{\vec{b}} \vec{E} \cdot d\vec{l} + \int_{\infty}^{\vec{a}} \vec{E} \cdot d\vec{l} \\ &= - \int_{\infty}^{\vec{b}} \vec{E} \cdot d\vec{l} - \int_{\vec{a}}^{\infty} \vec{E} \cdot d\vec{l} = - \int_{\vec{a}}^{\vec{b}} \vec{E} \cdot d\vec{l} \quad \dots (6) \end{aligned}$$

Form fundamental theorem of gradient



Hence, $\vec{E} = -\vec{\nabla}V$... (7)

We could have deduced it in other ways also. {Directly from vector calculus $\rightarrow$ As $\vec{\nabla} \times \vec{E} = 0$ and $\vec{\nabla} \cdot (\vec{\nabla}\phi) = 0$, ϕ be any scalar function- so $\vec{E} = \vec{\nabla}\phi$, for same scalar function ϕ , easily understood}.

Remember:

- (i) Potential and potential energy are different
- (ii) A surface over which the potential is constant is called an equipotential.
- (iii) Potential is a scalar quantity
- (iv) The reference point in general taken to be at infinity, but sometimes different reference points can also be chosen
- (v) Potential obeys the superposition principle.

$$V = V_1 + V_2 + \dots$$

- (vi) Unit of electric field $\rightarrow$ newton's/coulomb
- Unit of potential $\rightarrow$ newton-meters/coulomb
- or $\rightarrow$ joules/coulomb
- or $\rightarrow$ volt

Electric field of a point charge q at the origin is:

$$\vec{E} = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \hat{r}$$

We know that, $dV = -\vec{E} \cdot d\vec{\ell} = -\frac{1}{4\pi\epsilon_0} \frac{q}{r^2} r (dr \hat{r} + r d\theta \hat{\theta} + r \sin\theta d\phi \hat{\phi})$

$$\Rightarrow V = -\frac{q}{4\pi\epsilon_0} \int \frac{1}{r^2} dr = -\frac{q}{4\pi\epsilon_0} \left[\frac{1}{r} - \frac{1}{\infty} \right] = -\frac{1}{4\pi\epsilon_0} \frac{q}{r}$$

In general the potential of a point charge q at the distance r away from its position is

$$V(r) = \frac{1}{4\pi\epsilon_0} \frac{q}{r} \quad \dots(8)$$

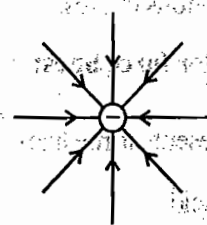
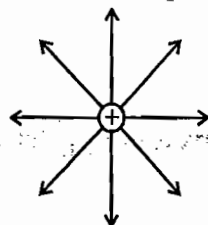
For a continuous charge distribution

$$V(r) = \frac{1}{4\pi\epsilon_0} \int \frac{1}{r} dq \quad \dots(9)$$

So what is the physical meaning of potential? It is the work done to move a unit positive charge from infinity to the point where the potential is to be measured.

Electric line of force:

An electric line of force is an imaginary curve drawn in such a way that tangent to this curve at any point gives the direction of electric field at the point.



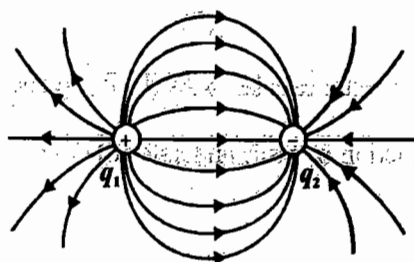


Figure (3)

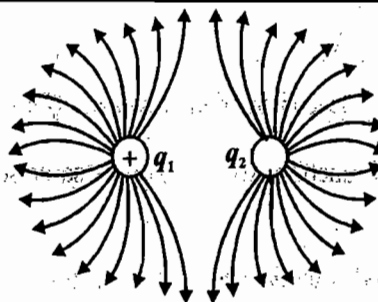


Figure (4)

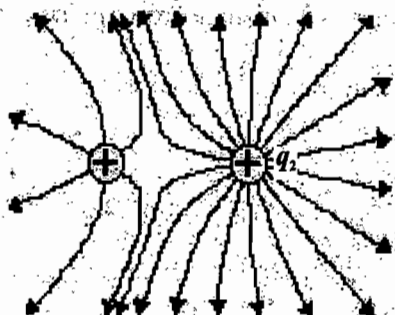


Figure (5)

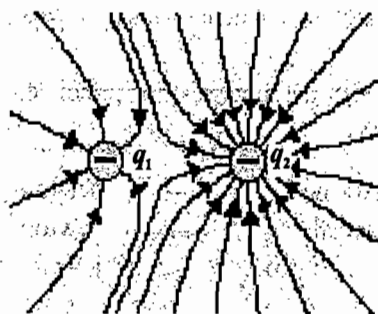


Figure (6)

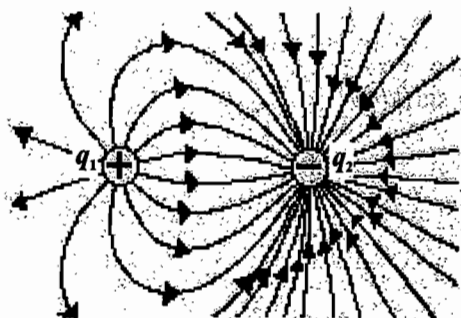


Figure (7)

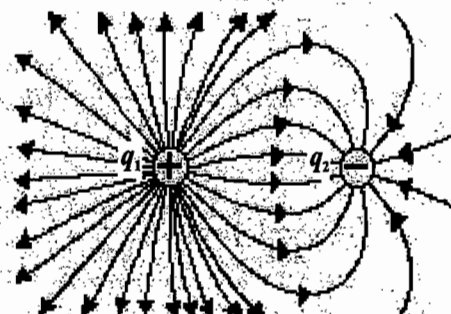


Figure (8)

- | | |
|---|--|
| (1) For single positive charge | (2) For single negative charge |
| (3) $q_1 = q_2$ and q_1 positive and q_2 negative. | (4) $q_1 = q_2$ and $q_1 =$ positive, $q_2 =$ positive |
| (5) $q_1 < q_2$ and both is positive | (6) $q_1 < q_2$ and both is negative |
| (7) $q_1 < q_2$ and q_1 is positive, $q_2 =$ negative | (8) $q_1 > q_2$ and q_1 is positive and q_2 is negative. |

Property of field lines:

- (1) Field lines start from a positive charge and end in a negative charge.
- (2) Since tangent of field line represent the direction of electric field and at a particular point the electric field do not have direction more than one. So, two field lines can not cross each other.
- (3) Fields lines are continuous curves.

Differential equation for force lines:

Suppose $\vec{r} = \vec{r}(s)$ represents force lines $\frac{d\vec{r}}{ds}$ is tangent to this curve at every point. Since tangent represent the direction of electric field.



$$\therefore \frac{dx}{ds} = aE_x, \frac{dy}{ds} = aE_y, \frac{dz}{ds} = aE_z$$

$$\therefore \frac{dx}{E_x} = \frac{dy}{E_y} = \frac{dz}{E_z} \quad \dots (10)$$

Gauss's law:

The total electric flux over any closed surface of any arbitrary electric field is equal to $\frac{1}{\epsilon_0}$ times the total charge enclosed by the surface.

For a closed surface 's' enclosing J number of point charge $q_1, \dots, q_j$, then the mathematical formula is

$$\oint \vec{E} \cdot d\vec{s} = \frac{1}{\epsilon_0} \sum_{i=1}^J q_i$$

Proof of Gauss's law:

Suppose a point charge q located at O and we consider a close surface enclosing q . Let we consider an elementary area $d\vec{s} = n ds$ at distance r from O .

Therefore, the electric field at P is

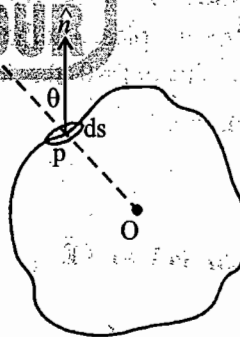
$$\vec{E} = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \hat{r}$$

Therefore, the flux through the $d\vec{s}$ is

$$\vec{E} \cdot d\vec{s} = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \hat{r} \cdot n ds$$

Therefore, total flux

$$\begin{aligned} \oint \vec{E} \cdot d\vec{s} &= \frac{q}{4\pi\epsilon_0} \int \frac{\hat{r} \cdot n}{r^2} ds \\ &= \frac{q}{4\pi\epsilon_0} \int ds \cos \theta \\ &= \frac{q}{4\pi\epsilon_0} 4\pi = \frac{q}{\epsilon_0} \end{aligned}$$



Poisson's and Laplace's equation from Gauss's law:

We know that Gauss's law is,

$$\oint \vec{E} \cdot d\vec{s} = \frac{q}{\epsilon_0}$$

For continuous charge distribution

$$\oint \vec{E} \cdot d\vec{s} = \int \frac{\rho dv}{\epsilon_0}$$

$$\oint (\nabla \cdot \vec{E}) dv = \int \frac{\rho dv}{\epsilon_0}$$



$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}$$

... (11)

This equation is the differential form of Gauss's law.

We know that, $E = -\nabla\phi$, put this in equation (11) we get

$$-\vec{\nabla} \cdot \vec{\nabla}\phi = \frac{\rho}{\epsilon_0} \Rightarrow \nabla^2\phi = -\frac{\rho}{\epsilon_0}$$

... (12)

This equation is called Poisson's equation.

If we put $\rho = 0$, the equation (12) become

$$\nabla^2\phi = 0$$

... (13)

is known as Laplace equation.

Application of Gauss's law:

Find the electric field for a uniformly charged infinite plane.

Consider a Gaussian cylindrical surface of crosssection area ΔS

$$\oint \vec{E} \cdot d\vec{S} = \frac{\sigma \Delta S}{\epsilon_0}$$

∴

$$\Rightarrow \vec{E} \cdot \Delta\vec{S} + \vec{E} \cdot \Delta\vec{S} = \frac{\sigma \Delta S}{\epsilon_0} \quad (\text{for both surface})$$

⇒

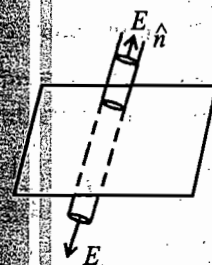
$$\Rightarrow \vec{E} = \frac{\sigma}{2\epsilon_0} \quad (\text{perpendicular direction of plane surface})$$

The charge remain outside the close surface:

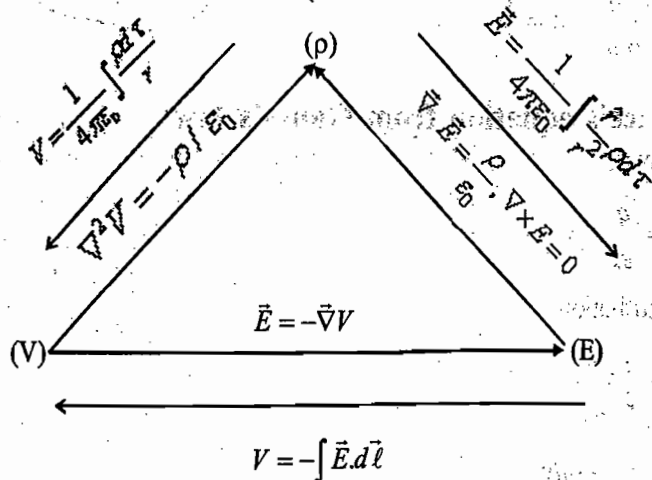
Suppose a point charge q is placed at the point P , which is outside the closed curve.

The net flux = flux in - flux out = 0 (flux in = flux out)

$$\therefore E \cdot \Delta S = 0$$



Relation between ρ , V and $\vec{E}$:





Example : A hemispherical surface of radius R is in a constant electric field $\vec{E}$ directed perpendicular to the base as shown in the figure. Determine the flux through the curved surface of the hemisphere.

Soln. The net flux through the closed surface is zero.

Flux through the curve surface:

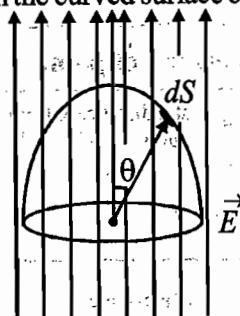
Let angle between E_0 and $\hat{n}$ is θ .

$$\phi_E = \int \vec{E} \cdot d\vec{S} = \int E \cos \theta dS$$

$$= \int_0^{\pi/2} \int_0^{2\pi} ER^2 \cos \theta \sin \theta d\theta d\phi$$

$$dS = R^2 \sin \theta d\theta d\phi$$

$$= \frac{1}{2} ER^2 \left(-\frac{\cos 2\theta}{2} \right)_0^{\pi/2} \cdot 2\pi = E\pi R^2$$



OR

$$\phi_{total} = \phi_{in} + \phi_{out} = 0 \quad (\text{for closed surface})$$

$$\therefore -\phi_{in} = \phi_{out}$$

$$\therefore \phi_{in} = \int \vec{E} \cdot d\vec{S} = E \cdot \pi R^2 \quad (\text{flux in through the circular surface})$$

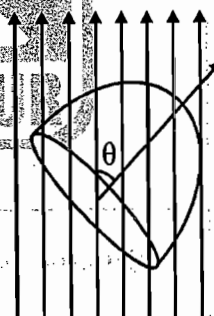
$$\therefore \phi_{out} = E \cdot \pi R^2 \quad (\text{flux out through the curve surface})$$

Special Case:

If θ be the angle between $\vec{E}$ and normal to the circular face.

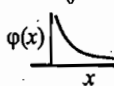
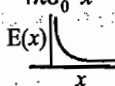
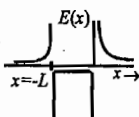
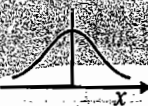
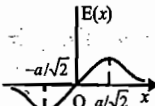
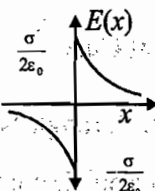
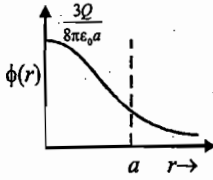
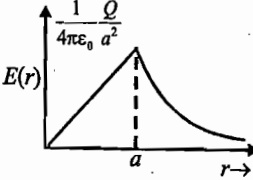
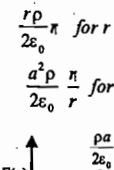
The flux out through the curve surface.

$$\phi = \cos \theta E \pi R^2$$





STANDARD TABLE OF FIELD & POTENTIAL

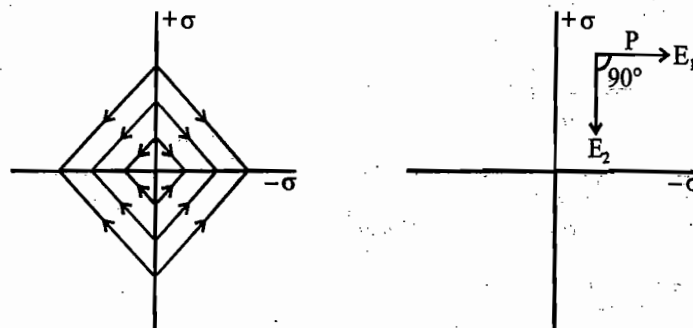
S.No.	Shape	Position of point	Potential	Field
(1)	Uniformly line charge distribution	 at a distance x from midpoint of line charge $x \gg L$	$\frac{1}{4\pi\epsilon_0} \frac{Q}{x}$ 	$\frac{1}{4\pi\epsilon_0} \frac{Q}{x^2}$ 
		at a distance x from the one end of	$\frac{1}{4\pi\epsilon_0} \frac{Q}{L} \ln\left(1 + \frac{L}{x}\right)$ 	$\frac{1}{4\pi\epsilon_0} \frac{Q}{x(L+x)}$ 
(2)	Uniformly charge ring	 on the axis at distance x from centre	$\frac{1}{4\pi\epsilon_0} \frac{Q}{\sqrt{a^2 + x^2}}$ 	$\frac{1}{4\pi\epsilon_0} \frac{Qx}{(a^2 + x^2)^{3/2}}$ 
(3)	Uniformly charged disc	 on the axial at distance x from centre	$\frac{\sigma}{2\epsilon_0} \left[\sqrt{a^2 + x^2} - x \right]$ 	$\frac{\sigma}{2\epsilon_0} \left[1 - \frac{x}{\sqrt{a^2 + x^2}} \right] \text{ for } x > 0$ $-\frac{\sigma}{2\epsilon_0} \left[1 + \frac{x}{\sqrt{a^2 + x^2}} \right] \text{ for } x < 0$ 
(4)	Uniformly charged sphere	 Outside the sphere  inside the sphere	$\frac{1}{4\pi\epsilon_0} \frac{Q}{r}$ $\frac{Q}{4\pi\epsilon_0 a} \left[\frac{3}{2} - \frac{r^2}{2a^2} \right]$ 	$\frac{1}{4\pi\epsilon_0} \frac{Q}{r^2} \text{ for } r > a$ $\frac{Q}{4\pi\epsilon_0 a^3} r \text{ for } r < a$ 
(5)	Uniformly charged long cylinder	 Inside the cylinder  Outside the cylinder	$\frac{a^2 \rho}{2\epsilon_0} \ln\left(\frac{r_0}{a}\right) + \frac{\rho}{4\epsilon_0} (a^2 - r^2)$ $\frac{a^2 \rho}{2\epsilon_0} \ln\left(\frac{r_0}{r}\right)$	$\frac{r \rho}{2\epsilon_0} \text{ for } r < a$ $\frac{a^2 \rho}{2\epsilon_0} \frac{\pi}{r} \text{ for } r > a$ 



Some examples of determination of electric field and potential:

1. Two uniformly charged infinite planes of charged density $+\sigma$ and $-\sigma$ intersect each other at right angles. Find the electric field at any point and hence sketch the field lines.

Solution :



Let any point P . The electric field due to planes are $E_1 = \frac{+\sigma}{2\epsilon_0}$ and $E_2 = \frac{-\sigma}{2\epsilon_0}$ and the angle between E_1 and E_2 is 90°

$$\therefore \text{Total field } E = \sqrt{E_1^2 + E_2^2}$$

$$= \sqrt{\left(\frac{\sigma}{2\epsilon_0}\right)^2 + \left(\frac{\sigma}{2\epsilon_0}\right)^2}$$

$$= \frac{\sqrt{2}\sigma}{2\epsilon_0} = \frac{\sigma}{\sqrt{2}\epsilon_0}$$

2. Find the electric field at a distance z above the midpoint of straight line segment of length $2L$, carrying uniform line charge λ .

Soln. Let us take an elementary length dx at a distance x from the centre. The field at the point 'P' due to elementary length is

$$d\vec{E} = \frac{1}{4\pi\epsilon_0} \frac{dx\lambda}{(z^2 + x^2)} \text{ along } \overline{OP}$$

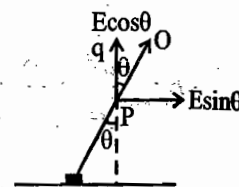
But it has two component $dE \sin \theta$ component will be cancelled out each other.

Therefore, the field along $\overline{pq}$ direction is

$$\vec{E} = \int dE \cos \theta = \int \frac{1}{4\pi\epsilon_0} \frac{dx\lambda}{(z^2 + x^2)} \cos \theta$$

$$= \frac{\lambda}{4\pi\epsilon_0} \int \frac{z \cos \theta \sec^2 \theta d\theta}{(x^2 + z^2)}$$

$$\Rightarrow \frac{\lambda}{4\pi\epsilon_0 z} \int \cos \theta d\theta = \frac{\lambda}{4\pi\epsilon_0 z} [\sin \theta]_{-\theta}^{\theta} = \frac{2\lambda}{4\pi\epsilon_0 z} \sin \theta = \frac{2\lambda}{4\pi\epsilon_0} \frac{L}{z\sqrt{z^2 + L^2}}$$



$$\tan \theta = \frac{x}{z}$$

$$dx = z \sec^2 \theta d\theta$$

$$\cos \theta = \frac{z}{\sqrt{(z^2 + x^2)}}$$



3. Find the potential and field along the axis of circular ring with uniform charge density λ .

Soln. Let consider as elementary length on the surface.

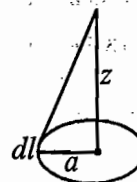
Therefore, the potential at the point P due to elementary length dl is

$$d\phi = \frac{1}{4\pi\epsilon_0} \frac{\lambda dl}{\sqrt{z^2 + a^2}}$$

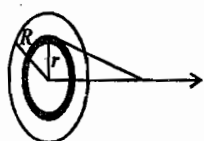
$$\therefore \phi = \frac{\lambda}{4\pi\epsilon_0} \frac{1}{\sqrt{z^2 + a^2}} \int dl$$

$$= \frac{\lambda}{4\pi\epsilon_0} \frac{2\pi a}{\sqrt{z^2 + a^2}} = \frac{1}{4\pi\epsilon_0} \frac{2\pi\lambda a}{\sqrt{z^2 + a^2}}$$

$$\therefore \vec{E} = -\frac{d\phi}{dz} = \frac{1}{4\pi\epsilon_0} \frac{2\pi\lambda a z}{(z^2 + a^2)^{3/2}} \hat{z}$$



4. The field and potential due to a circular surface with uniform surface charge density σ .



Soln. Let take an circular segment of radius r and $r + dr$.

The potential due to this element is

$$d\phi = \frac{1}{4\pi\epsilon_0} \frac{2\pi r dr \sigma}{\sqrt{r^2 + z^2}}$$

$$\phi = \frac{1}{4\pi\epsilon_0} 2\pi\sigma \int_0^R \frac{2r dr}{\sqrt{z^2 + r^2}} = \frac{\sigma}{2\epsilon_0} \left[\sqrt{z^2 + R^2} - |z| \right]$$

$$\therefore \vec{E} = -\frac{d\phi}{dz} = \frac{\sigma}{2\epsilon_0} \left(1 - \frac{z}{\sqrt{z^2 + R^2}} \right) \text{ for } z > 0$$

$$= -\frac{\sigma}{2\epsilon_0} \left(1 + \frac{z}{\sqrt{z^2 + R^2}} \right) \text{ for } z < 0$$

5. Two point charges $Q_1 = 1 \text{ nC}$ and $Q_2 = 2 \text{ nC}$ are kept in free space such that the distance between them is 0.1 m . [GATE 2001]

- The force on Q_2 is along the direction from Q_2 to Q_1
- The force on Q_2 is the same in magnitude as that on Q_1
- The force on Q_1 is attractive
- A point charge $Q_3 = -3 \text{ nC}$, placed at the midpoint between Q_1 and Q_2 , experiences no net force.

Soln. According to Coulomb law, the force on Q_2 due to Q_1 is

$$\vec{F}_{21} = \frac{1}{4\pi\epsilon_0} \frac{Q_1 Q_2}{r_{21}^2} \hat{r}_{21}$$



$$F_{12} = \frac{1}{4\pi\epsilon_0} \frac{Q_1 Q_2}{r_{12}^2} \hat{r}_{12}$$

$$|F_{12}| = |F_{21}|$$

Correct answer is (b)

6. If the electrostatic potential were given by $\phi = \phi_0 (x^2 + y^2 + z^2)$, where ϕ_0 is constant, then the charge density giving rise to the above potential would be : [GATE 2005]

- (a) 0 (b) $-6\phi_0 \epsilon_0$ (c) $-2\phi_0 \epsilon_0$ (d) $-\frac{6\phi_0}{\epsilon_0}$

Soln. We have, $\phi = \phi_0 (x^2 + y^2 + z^2)$

We know that Poisson's equation is

$$\nabla^2 \phi = -\frac{\rho}{\epsilon_0}$$

$\Rightarrow$

$$\rho = -\epsilon_0 \nabla^2 \phi = -\epsilon_0 \phi_0 (2+2+2) = -6\phi_0 \epsilon_0$$

Correct option is (b)

7. A solid sphere of radius R carries a uniform volume charge density ρ . The magnitude of electric field inside the sphere at a distance r from the centre is : [GATE 2005]

- (a) $\frac{r\rho}{3\epsilon_0}$ (b) $\frac{R\rho}{3\epsilon_0}$ (c) $\frac{R^2\rho}{r\epsilon_0}$ (d) $\frac{R^3\rho}{r^2\epsilon_0}$

Soln. We consider a Gaussian spherical surface of radius r.

$$\oint \vec{E} \cdot d\vec{s} = \frac{q_{\text{enclosed}}}{\epsilon_0}$$

$\Rightarrow$

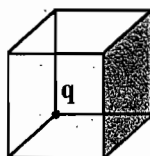
$$E \cdot 4\pi r^2 = \frac{\rho \cdot \frac{4}{3}\pi r^3}{\epsilon_0}$$

$\Rightarrow$

$$E = \frac{r\rho}{3\epsilon_0}$$

Correct option is (a)

8. A point charge 'q' sits at a corner of a cube of side 'a', as shown in the figure on the right. The flux of the electric field vector through the shaded side is [TIFR 2013]



- (a) $\frac{q}{8\epsilon_0}$ (b) $\frac{q}{16\epsilon_0}$ (c) $\frac{q}{24\epsilon_0}$ (d) $\frac{q}{6\epsilon_0}$

Soln. Let consider a Gaussian surface in shape of cube whose sides are $2a$. Now charge is at the centre



Therefore, flux through the area a^2 is $\frac{q}{24\epsilon_0}$

Correct option is (c)

9. A charge distribution has the charge density given by $\rho = Q\{\delta(x-x_0) - \delta(x+x_0)\}$. For this charge distribution the electric field at $(2x_0, 0, 0)$ [GATE 2013]

(a) $\frac{2Q\hat{x}}{9\pi\epsilon_0 x_0^2}$ (b) $\frac{Q\hat{x}}{4\pi\epsilon_0 x_0^3}$ (c) $\frac{Q\hat{x}}{4\pi\epsilon_0 x_0^2}$ (d) $\frac{Q\hat{x}}{16\pi\epsilon_0 x_0^2}$

Soln. We have one dimension charge density

$$\rho = Q\{\delta(x-x_0) - \delta(x+x_0)\}$$

Therefore, the total charge,

$$\begin{aligned} Q &= \int \rho dx = Q \int \delta(x-x_0) dx - Q \int \delta(x+x_0) dx \\ &= Q(\text{at } x=x_0) + (-Q)(\text{at } x=-x_0) \end{aligned}$$

Therefore, the field at $(2x_0, 0, 0)$ point is

$$\begin{aligned} \vec{E} &= \frac{1}{4\pi\epsilon_0} \left\{ \frac{Q}{x_0^2} - \frac{Q}{(3x_0)^2} \right\} \hat{x} \\ &= \frac{Q}{4\pi\epsilon_0} \left[\frac{9-1}{9x_0^2} \right] = \frac{2Q}{9\pi\epsilon_0 x_0^2} \hat{x} \end{aligned}$$

Correct option is (a)

10. Consider a particle of electric charge 'e' and mass 'm' moving under the influence of constant horizontal electric field E and constant vertical gravitational field described by acceleration due to gravity 'g'. If the particle starts from rest, what will be its trajectory? [JEST 2012]
- (a) Parabolic (b) elliptic (c) straight line (d) circular

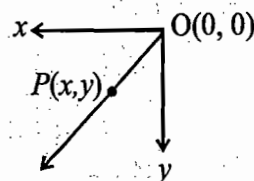
Soln. Let at $t=0$, electron at origine (0, 0) at $t=t$ the particle is at (x, y).

Acceleration along x-axis is $\frac{eE}{m}$ and acceleration along y-axis is g.

Therefore, the position of at a time t is $x = \frac{1}{2} \frac{eE}{m} t^2$ and $y = \frac{1}{2} g t^2$

$$\frac{y}{x} = \left(\frac{gm}{eE} \right)$$

$$y = \left(\frac{gm}{eE} \right) x$$



It is the equation of straight line.



11. If $\vec{E}_1 = xy\hat{i} + 2yz\hat{j} + 3xz\hat{k}$ and $\vec{E}_2 = y^2\hat{i} + (2xy + z^2)\hat{j} + 2yz\hat{k}$ then [JEST 2013]
- (a) Both are impossible electrostatic fields. (b) Both are possible electrostatic fields.
- (c) Only $\vec{E}_1$ is a possible electrostatic field. (d) Only $\vec{E}_2$ is a possible electrostatic field.

Soln. We have, $\vec{E}_1 = xy\hat{i} + 2yz\hat{j} + 3xz\hat{k}$

$$\vec{E}_2 = y^2\hat{i} + (2xy + z^2)\hat{j} + 2yz\hat{k}$$

$$\vec{\nabla} \times \vec{E}_1 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & 2yz & 3xz \end{vmatrix}$$

$$= \hat{i}(0 - 2y) + \hat{j}(0 - 3z) + \hat{k}(0 - x)$$

$$= -2y\hat{i} - 3z\hat{j} - x\hat{k} \neq 0$$

Therefore, $\vec{E}_1$ is not electrostatic field.

$$\vec{\nabla} \times \vec{E}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & (2xy + z^2) & 2yz \end{vmatrix}$$

$$= \hat{i}(2z - 2z) + \hat{j}(0 - 0) + \hat{k}(2y - 2y) = 0$$

Therefore, $\vec{E}_2$ is electrostatic field.

Correct option is (d).

12. For a scalar function ϕ satisfying the Laplace equation, $\nabla\phi$ has [GATE 2013]
- (a) zero curl and non-zero divergence (b) non-zero curl and zero divergence
- (c) zero curl and zero divergence (d) non-zero curl and non-zero divergence

Soln. If a scalar function ϕ satisfy the laplace equation then it obey the conditions

$$(i) \vec{\nabla} \times \vec{\nabla}\phi = 0$$

$$(ii) \vec{\nabla} \cdot \vec{\nabla}\phi = 0$$

Correct option is (c)

13. An electric field in a region is given by $\vec{E}(x, y, z) = ax\hat{i} + cz\hat{j} + 6by\hat{k}$. For which of a, b, c does this represent an electrostatic field? [JEST 2012]
- (a) 13, 1, 12 (b) 17, 6, 1 (c) 13, 1, 6 (d) 45, 6, 1

Soln. We have $\vec{E} = ax\hat{i} + cz\hat{j} + 6by\hat{k}$

For electrostatic field

$\vec{\nabla} \times \vec{E}$ should be zero.

$$\therefore \vec{\nabla} \times \vec{E} = 0$$



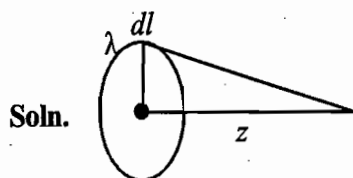
$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ ax & cz & 6by \end{vmatrix} = 0 \Rightarrow 6b = c$$

Only option (c) satisfy the condition.

Correct option is (c)

14. A circular loop of radius R , carries a uniform line charge density λ . The electric field, calculated at a distance z directly above the center of the loop, is maximum if z is equal to, [JEST 2015]

- (a) $\frac{R}{\sqrt{3}}$ (b) $\frac{R}{\sqrt{2}}$ (c) $\frac{R}{2}$ (d) $2R$



$$\phi = \frac{1}{4\pi\epsilon_0} \cdot \frac{\lambda dl}{\sqrt{R^2 + z^2}} = \frac{1}{4\pi\epsilon_0} \cdot \frac{\lambda 2\pi R}{\sqrt{R^2 + z^2}}$$

$$\therefore E = -\frac{\partial\phi}{\partial z} = \frac{\lambda}{2\epsilon_0} \frac{Rz}{(R^2 + z^2)^{3/2}}$$

For $\vec{E}$ max $\frac{\partial E}{\partial z} = 0$

$$\therefore \frac{1}{(R^2 + z^2)^{3/2}} - \frac{3}{2} \frac{2z^2}{(R^2 + z^2)^{5/2}} = 0 \Rightarrow (R^2 + z^2) - 3z^2 = 0 \Rightarrow z = \frac{R}{\sqrt{2}}$$

Correct option is (b)

15. The electrostatic potential due to a charge distribution is given by $V(r) = A \frac{e^{-\lambda r}}{r}$, where A and λ are constants. The total charge enclosed within a sphere of radius $1/\lambda$, with its origin at $r=0$ is given by,

- (a) $\frac{8\pi\epsilon_0 A}{e}$ (b) $\frac{4\pi\epsilon_0 A}{e}$ (c) $\frac{\pi\epsilon_0 A}{e}$ (d) 0 [JEST 2015]

Soln. We have, $V(r) = \frac{Ae^{-\lambda r}}{r}$

$$\therefore E = -\frac{dV}{dr} = -\hat{r} \frac{d}{dr} \left[\frac{e^{-\lambda r}}{r} \right] A = \left[\frac{\lambda e^{-\lambda r}}{r} + \frac{e^{-\lambda r}}{r^2} \right] A \hat{r}$$

According to Poisson's law,

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}$$



$$\begin{aligned}
 \Rightarrow \rho &= \epsilon_0 \vec{\nabla} \cdot \vec{E} = \epsilon_0 \nabla \cdot \left[\frac{\lambda e^{-\lambda r}}{r^2} \vec{r} + \frac{e^{-\lambda r}}{r^3} \vec{r} \right] A = \epsilon_0 \lambda A \nabla \cdot \left(\frac{e^{-\lambda r}}{r^2} \vec{r} \right) + \epsilon_0 A \nabla \cdot \left(\frac{e^{-\lambda r}}{r^3} \vec{r} \right) \\
 &= \epsilon_0 \lambda A e^{-\lambda r} \left(\vec{\nabla} \cdot \frac{\vec{r}}{r^2} \right) + \frac{\vec{r}}{r^2} \epsilon_0 \lambda A \cdot (\nabla e^{-\lambda r}) + \epsilon_0 A e^{-\lambda r} \left(\vec{\nabla} \cdot \frac{\vec{r}}{r^3} \right) + \left(A \epsilon_0 \frac{\vec{r}}{r^3} \cdot \vec{\nabla} e^{-\lambda r} \right) \\
 &= \epsilon_0 \lambda A e^{-\lambda r} \frac{1}{r^2} - A \epsilon_0 \lambda^2 \frac{1}{r} e^{-\lambda r} + A \epsilon_0 e^{-\lambda r} 4\pi \delta(r) - A \epsilon_0 \lambda \frac{1}{r^2} e^{-\lambda r} \\
 &= 4\pi A \epsilon_0 \left[\delta(r) - \frac{\lambda^2}{4\pi} \frac{1}{r} \right] e^{-\lambda r}
 \end{aligned}$$

$$\therefore q = 4\pi A \epsilon_0 \int \delta(r) e^{-\lambda r} dV - \frac{4\pi A \epsilon_0 \lambda^2}{4\pi} \int \frac{1}{r} e^{-\lambda r} 4\pi r^2 dr$$

$$\begin{aligned}
 &= 4\pi \epsilon_0 A - 4\pi \epsilon_0 A \int_0^1 e^{-\lambda z} dz \\
 &= 4\pi \epsilon_0 A - 4\pi \epsilon_0 A \left[-e^{-\lambda z} - e^{-\lambda} \right]_0^1 \\
 &= 4\pi \epsilon_0 A - 4\pi \epsilon_0 A \left[-e^{-1} - e^{-1} + 1 \right] = \frac{8\pi \epsilon_0 A}{e}
 \end{aligned}$$

Correct option is (a)





PRACTICE SET

- Electrostatic field is always
(a) solenoidal (b) harmonic in character
(c) irrotational (d) none of the above
- The unit of electric flux is
(a) gauss (b) $\text{N.m}^2.\text{C}^{-1}$ (c) weber (d) N.C^{-1}
- Work done in displacing a charge $2e$ through a distance of 0.1 m on an equipotential surface is
(a) 0.2 J (b) zero (c) 20 J (d) none of these
- If a proton, a deuteron and an α -particle are all accelerated through the same potential difference, their kinetic energies would be in the ratio of
(a) $1:2:4$ (b) $2:2:1$ (c) $1:1:1$ (d) $1:1:2$
- If the electric field and potential at a point are $\vec{E}$ and ϕ , respectively then
(a) for $\phi = 0$, $\vec{E}$ must be zero (b) for $E \neq 0$, $\phi \neq 0$
(c) for $\vec{E} = 0$, ϕ must be zero (d) none of the above
- An electric dipole is placed at the centre of a sphere. Then
(a) electric field is zero at every point on the sphere
(b) electric field is zero everywhere within the sphere
(c) electric field is zero outside the sphere
(d) total electric flux through the surface is zero
- An electron travelling in a region of electrostatic potential V_1 passes into a region of higher potential V_2 . The change in kinetic energy of the electron is proportional to
(a) $\sqrt{V_2 - V_1}$ (b) $(V_2 - V_1)$ (c) $V_2 - V_1$ (d) none of these
- The integral $\oint \vec{E} \cdot d\vec{\ell} = 0$ if the electric field $\vec{E}$ is caused by
(a) a static charge (b) both (a) and (b)
(c) a time varying magnetic field (d) none of these
- A sphere of radius R carries a volume charge density $\rho = \rho_0 r$, where ρ_0 is a constant and r is the radial distance from the centre of the charged sphere. The intensity of the electric field is proportional to:
(a) r^2 for $r \leq R$ and $1/r^2$ for $r \geq R$ (b) r for $r \leq R$ and $1/r^2$ for $r \geq R$
(c) $1/r^2$ for both $r \leq R$ and $r \geq R$ (d) $1/r$ for $r \leq R$ and $1/r^2$ for $r \geq R$
- A uniformly charged thin spherical shell of radius R carries a total charge q . At a point inside ($r < R$) the spherical shell, if E and V are intensity of the electric field and electric potential respectively then:
(a) Both E and V are zero.
(b) E is zero but V is a nonzero constant.
(c) E is proportional to $1/r^2$ and V is proportional to $1/r$.
(d) Both E and V are nonzero constants.

ANSWER KEY

Questions	1	2	3	4	5
Option	(c)	(b)	(b)	(d)	(d)
Questions	6	7	8	9	10

Chapter 3

Conductor and Polarization of Dielectrics

Introduction:

In previous chapter we have read electric field and potential due to static charge distribution in vacuum. In this chapter we will discuss the effect of electric field inside the conductor and dielectrics. Also we will read about electric dipole.

Conductor:

A conductor is a substances in which electrons can move freely like Au, Ag, Cu

Electrostatics of Conductors:

(1) Electric field inside the conductor is zero:

Since a perfect conductor have not any resistance, charge can move freely inside the conductor. As a result when we applied external electric field the induce charge distributed such a way so that the total external field canceled out by the internal field.

(2) Charge density inside the conductors is zero.

We know that, $\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0}$

Since, $E_{in} = 0$ Therefore, $\rho = 0$

(3) Free charges exist only on the surface of a conductor.

(4) Conductor forms an equipotential region.

Let consider two point on the conductor surface A and B.

We assume that the potential at A and B are $V(A)$, $V(B)$ respectively.

Therefore, the work done for remove a charge form A to B is given by

$$W = V(A) - V(B) = \int_B^A \vec{E} \cdot d\vec{\ell} = 0 \quad [\because E_{in} = 0]$$

$\Rightarrow$

$$\boxed{V(A) = V(B)}$$

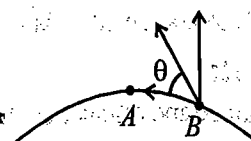
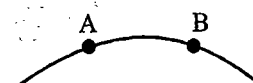
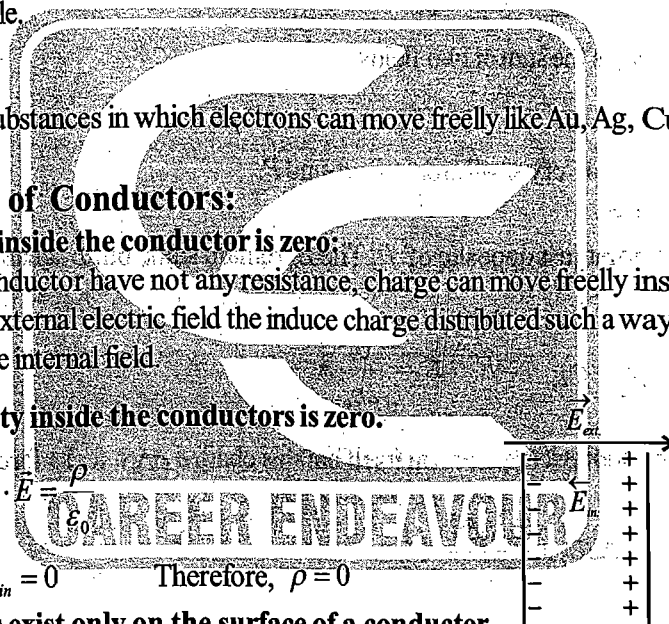
(5) The electric field outside of a conductor is normal to the surface.

Let $\vec{E}$ field is not normal to the surface, then it has a tangential component $E \cos \theta$. Now, any change bring from A to B.

Therefore, work done $\omega = E \cos \cdot AB$

Since, surface is equipotential, work doen is zero.

$$\vec{E} \cdot \vec{AB} \cos \theta = 0$$





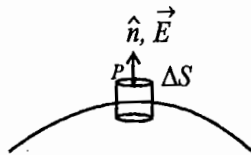
Therefore, the electric field is normal to the surface.

(6) Electric field strength just outside a conductor:

Let σ be the surface charge density on the surface and consider a cylindrical Gaussian surface whose cross section is ΔS .

$$\therefore \vec{E} \cdot \hat{n} \Delta S = \frac{\sigma \Delta S}{\epsilon_0}$$

$$\Rightarrow \boxed{\vec{E} = \frac{\sigma}{\epsilon_0} \hat{n}} \quad \dots (1)$$



(7) Mechanical Pressure :

Electric field at any external point P just outside the conductor is given by

$$\vec{E} = \frac{\sigma}{\epsilon_0} \hat{n}$$

This field may be considered as the sum of two fields.

$$\vec{E} = \vec{E}_1 + \vec{E}_2$$

Where E_1 is due to charges on a very small area ΔS around P.

E_2 is due to rest of the charges.

Let P' is internal point just opposite of P. The E_2 remain same but E_1 change such that total field become zero.

$$\text{Therefore, } 0 = -\vec{E}_1 + \vec{E}_2 \quad \text{Or, } \vec{E}_1 = \vec{E}_2 = \frac{\sigma}{2\epsilon_0} \hat{n}$$

The charge $\sigma \Delta S$ on ΔS cannot exert a force on itself but this charge $\sigma \Delta S$ feel a force due to $\vec{E}_2$.

$$\text{Therefore, the force is } \vec{F} = \sigma \Delta S \vec{E}_2 = \frac{\sigma \Delta S \hat{n}}{2\epsilon_0}$$

Therefore, pressure,

$$P = \frac{\vec{F}}{\Delta S \hat{n}} = \frac{\sigma^2}{2\epsilon_0}$$

$$\boxed{P = \frac{\sigma^2}{2\epsilon_0}} \quad \dots (2)$$

$$\boxed{P = \frac{1}{2} \epsilon_0 E_0^2} \quad \dots (3)$$

Pressure act on surface normally outward direction.

Electric dipole:

Two equal and opposite charge keep at a small distance is called electric dipole.

Electric dipole moment:

The multiple of one of it charge and distance between two point charges is called dipole moment. The direction of dipole moment is from negative charge to positive.

The potential due to dipole:

Let we consider a dipole whose position of +q charge is $\vec{r}_1$ and position of -q charge $\vec{r}_1 + \vec{\ell}$.

Now the potential at the point $P(\vec{r})$ is given by

$$V(r) = \frac{1}{4\pi\epsilon_0} \left[\frac{q}{|\vec{r} - \vec{r}_1|} + \frac{q}{|\vec{r} - \vec{r}_1 - \vec{d}|} \right]$$

$$= \frac{1}{4\pi\epsilon_0} \left[\frac{q}{R} + \frac{q}{|\vec{R} - \vec{d}|} \right]$$

$$V(r) = \frac{q}{4\pi\epsilon_0} \left[\frac{1}{R} + \frac{1}{|\vec{R} - \vec{d}|} \right]$$

$$= \frac{q}{4\pi\epsilon_0} \left[\frac{1}{R} + \frac{1}{R} + \frac{Rd}{R^3} \right]$$

(since $d \ll R$)

$$= \frac{q}{4\pi\epsilon_0} \left[\frac{Rd}{R^3} \right] = \frac{1}{4\pi\epsilon_0} \left[\frac{\vec{R} \cdot \vec{p}}{R^3} \right]$$

$$V(r) = \frac{1}{4\pi\epsilon_0} \left(\frac{\vec{p} \cdot \vec{R}}{R^3} \right)$$

... (5)

If negative at origin and charges are lie along z-axis and the position of the point P is (r, θ) then

$$V(r) = \frac{1}{4\pi\epsilon_0} \left(\frac{\vec{p} \cdot \vec{r}}{r^3} \right)$$

Or,

$$V(r) = \frac{1}{4\pi\epsilon_0} \frac{pr \cos \theta}{r^3}$$

Or,

$$V(r) = \frac{1}{4\pi\epsilon_0} \frac{p \cos \theta}{r^2}$$

... (6)

The electric field due to dipole:

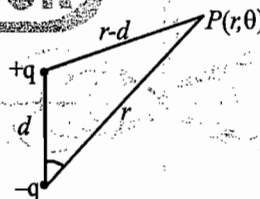
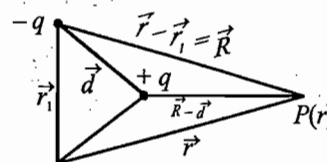
We know that,

$$\vec{E} = -\vec{\nabla} V = -\vec{\nabla} \left(\frac{1}{4\pi\epsilon_0} \left(\frac{\vec{p} \cdot \vec{R}}{R^3} \right) \right) \text{ (using equation (5))}$$

$$= -\frac{1}{4\pi\epsilon_0} \left[\frac{1}{R^3} \nabla (\vec{p} \cdot \vec{R}) + \vec{p} \cdot \vec{R} \left(\vec{\nabla} \frac{1}{R^3} \right) \right]$$

$$\vec{E}(r) = \frac{1}{4\pi\epsilon_0} \left[3 \frac{(\vec{p} \cdot \vec{R}) \vec{R}}{R^5} - \frac{\vec{p}}{R^3} \right]$$

... (7)



If the negative charge at origin then

$$\begin{aligned}\vec{E}(r) &= -\vec{\nabla}V(r) = -\vec{\nabla} \frac{1}{4\pi\epsilon_0} \left(\frac{p \cos \theta}{r^2} \right) \\ &= -\frac{p}{4\pi\epsilon_0} \left[\hat{r} \frac{\partial}{\partial r} + \hat{\theta} \frac{1}{r} \frac{\partial}{\partial \theta} + \hat{\phi} \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} \right] \left(\frac{\cos \theta}{r^2} \right) \\ &= \frac{p}{4\pi\epsilon_0} \frac{2 \cos \theta}{r^3} \hat{r} + \frac{p}{4\pi\epsilon_0} \frac{\sin \theta}{r^3} \hat{\theta} \\ &= E_r \hat{r} + E_\theta \hat{\theta}\end{aligned}$$

where, $E_r = \frac{1}{4\pi\epsilon_0} \frac{2p \cos \theta}{r^3}$, $E_\theta = \frac{1}{4\pi\epsilon_0} \frac{p \sin \theta}{r^3}$

$$|\vec{E}| = \sqrt{E_r^2 + E_\theta^2} = \frac{1}{4\pi\epsilon_0} \frac{p}{r^3} \sqrt{4 \cos^2 \theta + \sin^2 \theta} = \frac{1}{4\pi\epsilon_0} \frac{p}{r^3} \sqrt{3 \cos^2 \theta + 1}$$

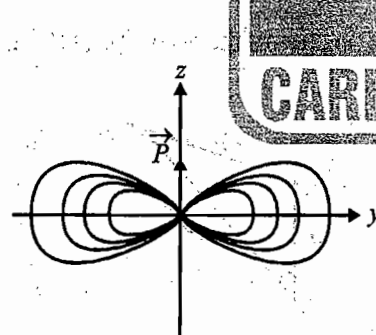
Equipotential surface and field lines of dipole:

If θ is the angle between dipole and r then the equation of field lines is

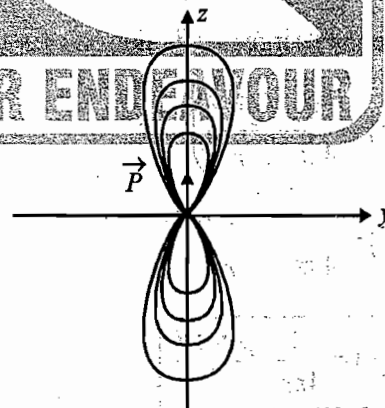
$$r = c \sin^2 \theta \quad (\text{where, } c \text{ is a constant}) \quad (8.a)$$

The equation of equipotential line

$$r(\theta) = A \sqrt{\cos \theta} \quad (\text{where, } A \text{ is a constant}) \quad (8.b)$$



Field line of ideal dipole



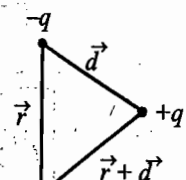
Equipotential line of ideal dipole

Potential energy of a dipole placed in an external electric field:

Let we put a dipole whose $-q$ and $+q$ respectively at $\vec{r}$ and $(\vec{r} + \vec{d})$

The potential energy,

$$\begin{aligned}U &= -qV(\vec{r}) + qV(\vec{r} + \vec{d}) \\ U &= -qV(r) + qV(r) + q\vec{d} \cdot \vec{\nabla}V(r) \quad (\text{using Taylor expansion}) \\ &= q\vec{d} \cdot \vec{\nabla}V(r)\end{aligned}$$





$$U = \vec{p} \cdot \vec{\nabla} V(r) \quad \dots (9)$$

$$\text{Or, } U = -\vec{p} \cdot \vec{E}(r) \quad (\because E(r) = -\nabla V(r)) \quad \dots (10)$$

Force and torque on a dipole placed in an electric field :

We know that, $\vec{F} = -\vec{\nabla} U = \nabla(\vec{p} \cdot \vec{E})$

$$\begin{aligned} &= (\vec{p} \cdot \vec{\nabla}) \vec{E} + (\vec{E} \cdot \vec{\nabla}) \vec{p} + \vec{p} \times (\vec{\nabla} \times \vec{E}) + \vec{E} \times (\vec{\nabla} \times \vec{p}) \\ &= (\vec{p} \cdot \vec{\nabla}) \vec{E} + 0 + 0 + 0 \end{aligned}$$

$$\vec{F} = (\vec{p} \cdot \vec{\nabla}) \vec{E} \quad \dots (11)$$

For uniform $\vec{E}$,

$$\vec{F} = (\vec{p} \cdot \vec{\nabla}) \vec{E} = 0$$

that means there have no translational motion.

Torque on the dipole in uniform E field :

We know the potential energy of the dipole due external electric field is $u = \vec{p} \cdot \vec{E} = pE \cos \theta$

Therefore, the torque about its own centre is

$$\tau = -\frac{du}{d\theta} = pE \sin \theta$$

$$\text{Or, } \vec{\tau} = \vec{p} \times \vec{E}$$

Torque on the dipole in non-uniform electric field :

The torque on a small dipole about its own centre is still

$$\vec{\tau} = \vec{p} \times \vec{E}$$

But since there is a net force $\vec{F}$ on the dipole.

Therefore, an extra torque act on the dipole is $\vec{r} \times \vec{F}$ about any other point.

Therefore, total torque about any other point is $\vec{\tau} = \vec{p} \times \vec{E} + \vec{r} \times \vec{F}$

Mutual potential energy between two coplanar dipoles:

Let consider two dipole of dipole moment $\vec{p}_1$ and $\vec{p}_2$ lying on a single plane, $\vec{r}_{21}$ is the position vector of dipole 2 with respect to 1.

The electric field at the centre of dipole 2 is

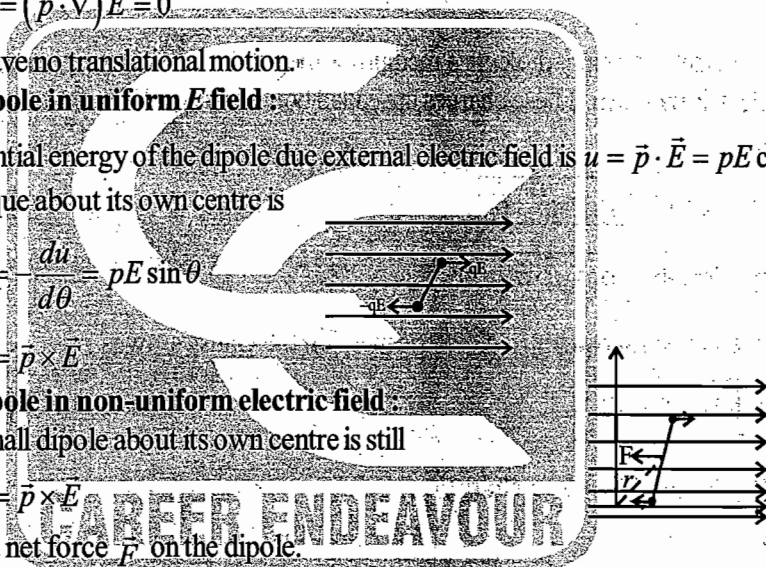
$$\vec{E}_1 = \frac{1}{4\pi\epsilon_0} \left[\frac{3(\vec{p}_1 \cdot \vec{r}_{21})\vec{r}_{21}}{r_{21}^5} - \frac{\vec{p}_1}{r_{21}^3} \right]$$

The mutual potential energy,

$$U_{21} = -\vec{p}_2 \cdot \vec{E}_1 = -\vec{p}_1 \cdot \vec{E}_2 = \frac{1}{4\pi\epsilon_0} \left[\frac{\vec{p}_1 \cdot \vec{p}_2}{r_{21}^3} - \frac{3(\vec{p}_1 \cdot \vec{r}_{21})(\vec{p}_2 \cdot \vec{r}_{21})}{r_{21}^5} \right] \dots (13)$$

Polar and non-polar molecules:

Polar: Centre of positive and negative charge do not coincide this type molecules are called polar molecules like H_2O , H_2S .





Dielectric polarization :

When any dielectric is placed in an electrostatic field the charge distribution within the molecules may become modified. The electrons are pulled in one direction and the nuclei in the opposite way. This causes a slight displacement of the centre of positive and negative charges. This phenomenon is known as electric polarization.

In case of polar dielectric in the absence of external electric field the dipoles are randomly oriented due to thermal agitation and the net dipole moment per unit volume is zero. If an electric field is applied it tries to align the dipoles along the field direction while thermal agitation tries to oppose the alignment. In the equilibrium some net alignment hence some net dipole moment per unit volume is obtained. This phenomenon is known as orientational polarization.

Polarization $\vec{P}$:

The electric dipole moment per unit volume is called polarization.

$$\vec{P} = \lim_{\Delta V \rightarrow 0} \frac{1}{\Delta V} \sum_i \vec{p}_i$$

Bound charge :

Due to polarization of any dielectric medium some kind of charges appear in the volume and on the surface of the dielectric medium. These charges are called bound charge. When external field vanishes the bound charge all vanish.

The volume charge density, $\rho_b = -\vec{\nabla} \cdot \vec{P}$

The surface charge density, $\sigma_b = \vec{P} \cdot \hat{n}$

Gauss's law in dielectric medium: In dielectric medium the Gauss's law is

$$\oint \vec{D} \cdot d\vec{S} = Q_f \quad (14)$$

Where, $\vec{D} = \epsilon_0 \vec{E} + \vec{P}$ and Q_f = free charge
 $= \epsilon \vec{E}$

Poisson's and Laplace's equations in dielectrics:

We know $\oint \vec{D} \cdot d\vec{S} = Q_f$

$$\Rightarrow \int (\vec{\nabla} \cdot \vec{D}) dv = \int \rho_f dv \Rightarrow \vec{\nabla} \cdot \vec{D} = \rho_f \Rightarrow \vec{\nabla} \cdot \vec{E} = \frac{\rho_f}{\epsilon}$$

$$\Rightarrow -\vec{\nabla}^2 \phi = \frac{\rho_f}{\epsilon_0} \quad (\text{using } \vec{E} = -\vec{\nabla} \phi)$$

$$\Rightarrow \boxed{\nabla^2 \phi = -\frac{\rho_f}{\epsilon_0}} \text{ is known as Poisson's equation.} \quad \dots (15)$$

If medium is charge free then $\rho_f = 0$

$$\therefore \boxed{\nabla^2 \phi = 0} \text{ is known as Laplace equation.} \quad \dots (16)$$

Susceptibility, permittivity and dielectric constant:

If the external electric field is not very strong, then



$$\begin{aligned}\therefore \vec{D} &= \epsilon_0 \vec{E} + \vec{P} \\ &= \epsilon_0 \vec{E} + \epsilon_0 \chi \vec{E} = \epsilon_0 (1 + \chi) \vec{E} = \epsilon \vec{E}\end{aligned}$$

where, $\epsilon = \epsilon_0 (1 + \chi)$

$$\Rightarrow \frac{\epsilon}{\epsilon_0} = (1 + \chi) = k \text{ is called dielectric constant.} \quad \dots (17)$$

For isotropic medium χ is scalar, but for anisotropic medium it is a tensor.

$$\begin{pmatrix} P_x \\ P_y \\ P_z \end{pmatrix} = \epsilon_0 \begin{pmatrix} \chi_{xx} & \chi_{xy} & \chi_{xz} \\ \chi_{xy} & \chi_{yy} & \chi_{yz} \\ \chi_{xz} & \chi_{yz} & \chi_{zz} \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}$$

Non-polar liquid dielectric and the Clausius-Mossotti Relation:

The electric field that is responsible for polarizing a molecule of the dielectric is called the molecular field or local field. For simple non-polar liquid dielectrics we may expect that a molecule finds itself, on the average, within a spherical hole. To calculate the molecular field at a point, let us imagine to remove a small sphere of polarized material from the region surrounding the point under consideration. Let $\vec{E}$ be the uniform macroscopic field within the dielectric before the sphere was removed. By the superposition principle $\vec{E}$ may be considered as the sum of the field $\vec{E}_1$ within the cavity due to all charges outside the cavity and the field $\vec{E}_2$ due to all charges within the removed polarized sphere, i.e.

$$\vec{E} = \vec{E}_1 + \vec{E}_2 \quad \dots (18)$$

To find $\vec{E}_2$ we consider a uniformly polarized sphere as shown in figure. Consider an elementary area $d\vec{S} = \hat{n}dS$ at any point $A(r, \theta, \phi)$ on the surface of the sphere and take the z-axis along the direction of polarization $\vec{P}$.

The surface density of polarization charge at A is $\vec{P} \cdot \hat{n} = P \cos \theta$. The field at O due to charges on dS is

$$d\vec{E}_2 = \frac{P \cos \theta dS}{4\pi\epsilon_0 r^2} \hat{r} \quad \dots (19)$$

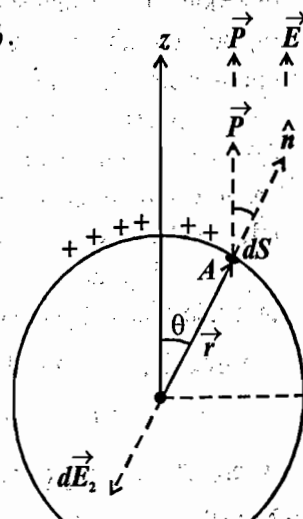
It is along the direction $\vec{AO}$.

Because of symmetry about z-axis only the component of $d\vec{E}_2$ along the negative z-axis will contribute to the integral of equation (19) over the complete surface, the other component perpendicular to z-axis will give zero contribution when integrated over the whole surface. Since, $dS = r^2 \sin \theta d\theta d\phi$.

$$\begin{aligned}\vec{E}_2 &= -\frac{1}{4\pi\epsilon_0} \int \frac{P \cos \theta dS}{r^2} \cos \theta \hat{z} \\ &= -\frac{\vec{P}}{4\pi\epsilon_0} \int_0^{2\pi} d\phi \int_0^\pi \cos^2 \theta \sin \theta d\theta \quad [\because \vec{P} = P\hat{z}] \\ &= -\frac{\vec{P}}{4\pi\epsilon_0} \cdot 2\pi \cdot \frac{2}{3} = -\frac{\vec{P}}{3\epsilon_0} \quad \dots (20)\end{aligned}$$

Thus, the field inside is uniform. It is a depolarizing field.

Now from equation (18) we get the field $\vec{E}_1$ within the cavity as





For non-polar dielectrics with simple structures such as for most dielectrics liquids, the molecular field that polarizes a molecule may be taken as $\vec{E}_1$. Under the action of this field the molecule will be polarized. The dipole moment $\vec{p}$ induced in the molecule will be proportional to $\vec{E}_1$, i.e.,

$$\vec{p} = \sigma \vec{E}_1 \quad \dots (22)$$

where the constant of proportionality α is called the molecular polarizability. It is the dipole moment per unit polarizing field. Its value depends on the detailed structure of the molecule in question. If there are n molecules per unit volume of the dielectric then the polarization vector.

$$\vec{P} = n\vec{p} = n\alpha \vec{E}_1 = n\alpha \left(\vec{E} + \frac{\vec{P}}{3\epsilon_0} \right) \quad [\text{using equation (21)}]$$

$$\text{Or,} \quad \vec{P} = \frac{n\alpha}{1 - n\alpha / 3\epsilon_0} \vec{E} \quad \dots (23)$$

This equation may be written in terms of the dielectric constant K of the medium. We have the relation

$$\vec{D} = \epsilon_0 \vec{E} + \vec{P} \quad \text{or} \quad \vec{P} = \epsilon \vec{E} - \epsilon_0 \vec{E} = \epsilon_0 (K - 1) \vec{E} \quad \left[K = \frac{\epsilon}{\epsilon_0} \right]$$

Now putting this value of $\vec{P}$ in equation (23) we get

$$\epsilon_0 (K - 1) = \frac{n\alpha}{1 - \frac{n\alpha}{3\epsilon_0}}$$

$$\text{Or,} \quad \alpha = \frac{3\epsilon_0}{n} \cdot \frac{K - 1}{K + 2} \quad \dots (24)$$

This is known as Clausius-Mossotti equation.

Polar Dielectrics and the Langevin-Debye formula:

Let us consider a polar gaseous or liquid dielectric in which each molecule has a permanent dipole moment $\vec{p}$. In the absence of any electric field the dipoles because of random thermal agitation, point in random directions and net dipole moment per unit volume is zero. When an electric field is applied two things happen

(a) Individual dipoles experiences torques, which tend to align them along the field direction. Thermal agitation opposes the alignment and at ordinary temperatures and electric fields the alignment is far from complete. But there is some net alignment and so some net dipole moment per unit volume results.

(b) There is some extra dipole moment induced due to increases in separation between positive and negative charge centres as in the case of non-polar molecules. We shall neglect this second effect for the moment but its effect will be added in later.

This polarization that occurs due to orientation effect can be calculated as follows. According to a principle of statistical mechanics, in thermal equilibrium, the probability of finding a molecule with potential energy U is proportional to $e^{-U/kT}$, where k is the Boltzmann constant, $\vec{p}$ is the dipole moment and $\vec{E}$ is the electric field.

$$U = -\vec{p} \cdot \vec{E} = -pE \cos \theta \quad \dots (25)$$

where, θ is the angle between $\vec{p}$ and $\vec{E}$



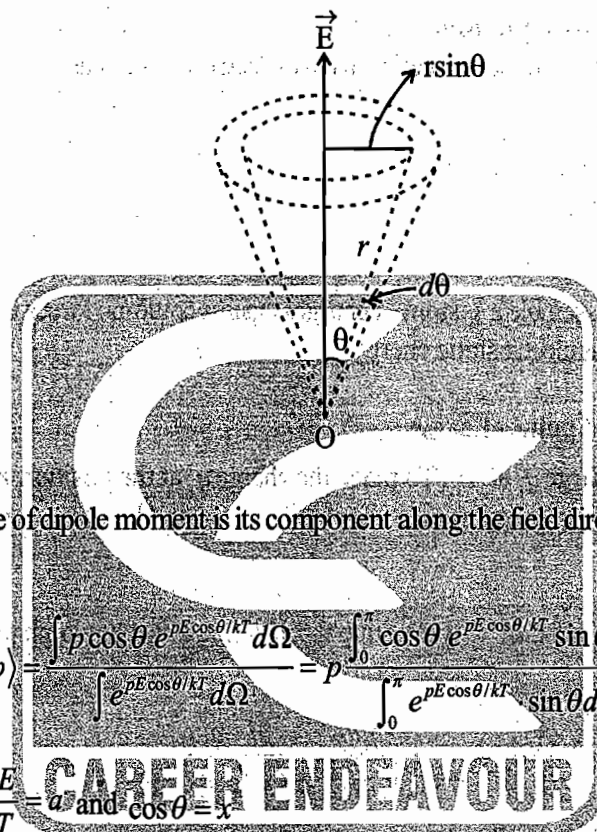
field that
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So, the probability that a dipole will have orientation θ with respect to the field is proportional to $e^{pE \cos \theta / kT}$. The probability that the axis of a dipole lies between the angles θ and $\theta + d\theta$, i.e. within the conical solid angle,

$$d\Omega = \frac{2\pi r \sin \theta \cdot r d\theta}{r^2} = 2\pi \sin \theta d\theta \quad \dots (26)$$

t per unit
olecules

is proportional to $e^{pE \cos \theta / kT} d\Omega$



The effective value of dipole moment is its component along the field direction, i.e., $p \cos \theta$. Now its average value is

$$\langle p \rangle = \frac{\int p \cos \theta e^{pE \cos \theta / kT} d\Omega}{\int e^{pE \cos \theta / kT} d\Omega} = p \frac{\int_0^\pi \cos \theta e^{pE \cos \theta / kT} \sin \theta d\theta}{\int_0^\pi e^{pE \cos \theta / kT} \sin \theta d\theta} \quad \dots (27)$$

Putting,

$$\frac{pE}{kT} = a \text{ and } \cos \theta = x$$

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We get

$$\langle p \rangle = p \frac{\int_1^{-1} x e^{ax} dx}{\int_1^{-1} e^{ax} dx} = p \frac{\left[\frac{x e^{ax}}{a} - \frac{e^{ax}}{a^2} \right]_1^{-1}}{\left[\frac{e^{ax}}{a} \right]_1^{-1}} = p \left[\frac{e^a + e^{-a}}{e^a - e^{-a}} - \frac{1}{a} \right] = p \left[\coth a - \frac{1}{a} \right]$$

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t but its

$$\therefore \frac{\langle p \rangle}{p} = \coth a - \frac{1}{a} \quad \dots (28)$$

ing to a
otential

$$\text{where, } a = \frac{pE}{kT}$$

(1) At large electric field strengths or low temperature, $a = \frac{pE}{kT} \gg 1$, then $\frac{\langle p \rangle}{p} = 1$ which means that complete alignment of dipoles.

(2) When $a \ll 1$ at room temperature and weak electric field



$$\langle p \rangle = \frac{p^2 E}{3kT} \quad \dots (29)$$

In deriving this formula we have considered only the orientational polarization. For this α is called orientational polarizability. If we consider deformational or electronic polarizability in addition to the orientational effect then the total molecular polarizability becomes

$$\alpha = \alpha_0 + \frac{p^2}{3kT} \quad \dots (30)$$

This is known as **Langevin-Debye equation**.

Using equation (13), we can write equation (7) in terms of dielectric constant as

$$\frac{K-1}{K+2} = \frac{n}{3\epsilon_0} \left(\alpha_0 + \frac{p^2}{3kT} \right) \quad \dots (31)$$

Electrostatic Energy:

The work necessary to assemble a system of charges against coulomb forces is stored in the system as potential energy. This is known as electrostatic energy.

Electrostatic energy system of point charges:

Let at any time place a point charge q_1 at r_1 . To place the charge q_1 at the position r_1 , we require no work because there is no interacting coulomb field.

$$\therefore U_1 = 0$$

But to bring the charge q_2 to the position r_2 , we require work done against coulomb repulsion due to q_1 .

$$\text{This work } U_2 = -\frac{1}{4\pi\epsilon_0} \int_{\infty}^{r_2} \frac{q_1 q_2}{r^2} dr = \frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{r_1 - r_2} = \frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{r_{12}}$$

To bring another charge, we need work done.

$$U_3 = \frac{1}{4\pi\epsilon_0} \left[\frac{q_1 q_3}{r_{13}} + \frac{q_2 q_3}{r_{23}} \right]$$

Therefore, net work done

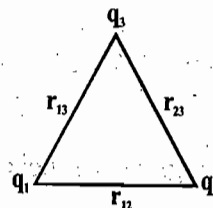
$$U = U_1 + U_2 + U_3 = 0 + \frac{1}{4\pi\epsilon_0} \left[\frac{q_1 q_2}{r_{12}} + \frac{q_1 q_3}{r_{13}} + \frac{q_2 q_3}{r_{23}} \right]$$

For N point charged,

$$U = \frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N \frac{1}{4\pi\epsilon_0} \frac{q_i q_j}{r_{ij}} \quad \dots (32)$$

We know electrostatic potential

$$\phi_i = \sum_{\substack{j=1 \\ i \neq j}}^N \frac{1}{4\pi\epsilon_0} \frac{q_j}{r_{ij}}$$



**For continuous charge distribution :**

For continuous charge distribution u is given by

$$U = \frac{1}{2} \int \rho(\vec{r}) \phi(\vec{r}) d\tau \quad [\rho(\vec{r}) \text{ is volume charge distribution}] \quad \dots (35)$$

Electrostatic Energy in terms of field :

Suppose we have a finite region of space V in a dielectric medium of permittivity ϵ and the volume charge density ρ .

Therefore, the electrostatic energy of the system is given by

$$U = \frac{1}{2} \int \rho(r) \phi(r) dV = \frac{1}{2} \int (\nabla \cdot \vec{D}) \phi dV \quad [\text{Gauss's law } \nabla \cdot \vec{D} = \rho]$$

$$= \frac{1}{2} \int (\vec{\nabla} \cdot \phi \vec{D}) - \vec{\nabla} \phi \cdot \vec{D} dV$$

$$= \frac{1}{2} \int (\nabla \cdot \phi \vec{D}) dV - \frac{1}{2} \int (\nabla \phi \cdot \vec{D}) dV$$

$$= \frac{1}{2} \int (\phi \vec{D}) d\vec{S} + \frac{1}{2} \int (\vec{E} \cdot \vec{D}) dV$$

At large distance from the charge distribution the first term will be vanish.

$$\therefore U = \frac{1}{2} \int (\vec{E} \cdot \vec{D}) dV \quad \dots (35)$$

Therefore, the energy density, $u = \frac{1}{2} (\vec{E} \cdot \vec{D}) \quad \dots (36)$

Cases:

(i) For isotropic dielectric medium, $\vec{D} = \epsilon \vec{E}$, $u = \frac{1}{2} \epsilon E^2$

(ii) For free space, $\vec{D} = \epsilon_0 \vec{E}$, $u = \frac{1}{2} \epsilon_0 E^2$

Forces and Torques from electrostatic energy:

We know work done = force . displacement

$$\Rightarrow dW = \vec{F} \cdot d\vec{r}$$

Also for isolated system, the work done

$$dW = -dU = -\vec{\nabla} U \cdot d\vec{r}$$

$$\therefore \vec{F} = -\vec{\nabla} U \quad \dots (37)$$

If the part of the system under consideration is constrained to rotate about an axis and it undergoes a small rotation $d\theta$ under the action of a torque $\vec{\tau}$ then,

$$dW = \vec{\tau} \cdot d\vec{\theta} = -dU = -\vec{\nabla}_\theta U \cdot d\vec{\theta}$$

$$\therefore \vec{\tau} = -\vec{\nabla}_\theta U \quad \dots (38)$$



Suppose AB be the boundary of two medium of dielectric constants ϵ_1 and ϵ_2 . Now considers an Gaussian cylinder of cross-section ΔS .

Therefore, according to Gauss's law

$$\oint \vec{D} \cdot d\vec{s} = \sigma_{free} \cdot \Delta S$$

$$\Rightarrow \vec{D}_1 \cdot \Delta \vec{s} - \vec{D}_2 \cdot \Delta \vec{s} = \sigma_{free} \cdot \Delta S$$

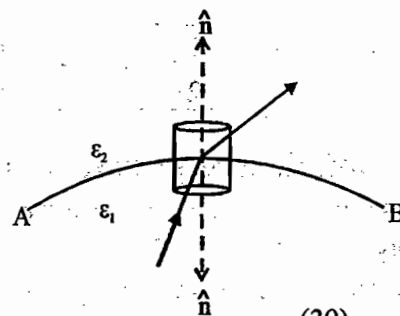
$$\Rightarrow \vec{D}_2 \cdot \hat{n} - \vec{D}_1 \cdot \hat{n} = \sigma_{free}$$

$$\Rightarrow \epsilon_2 E_{2n} - \epsilon_1 E_{1n} = \sigma_{free}$$

If the boundary is charge free then

$$\epsilon_2 E_{2n} = \epsilon_1 E_{1n}$$

... (39)



Now, consider rectangular path 1234

$$\oint (\vec{\nabla} \times \vec{E}) \cdot d\vec{s} = \oint \vec{E} \cdot d\vec{\ell} = 0$$

$$\Rightarrow E_1 \cdot \Delta \ell - E_2 \cdot \Delta \ell = 0$$

$$\Rightarrow E_1 \cdot \Delta \ell = E_2 \cdot \Delta \ell$$

$$\Rightarrow E_{2t} = E_{1t}$$

... (41)

SOLVED PROBLEMS

1. A conducting sphere of radius a carries a charge q . It is coated with a linear dielectric material of permittivity ϵ out to a radius b . Find the potential at the centre of the conductor.

Soln. Applying Gauss's law to a spherical surface of radius r , where $a < r < b$

$$\oint \vec{D} \cdot d\vec{s} = q$$

$$\Rightarrow D \cdot 4\pi r^2 = q$$

$$\Rightarrow E = \frac{1}{4\pi\epsilon} \frac{q}{r^2} \text{ for } a < r < b$$

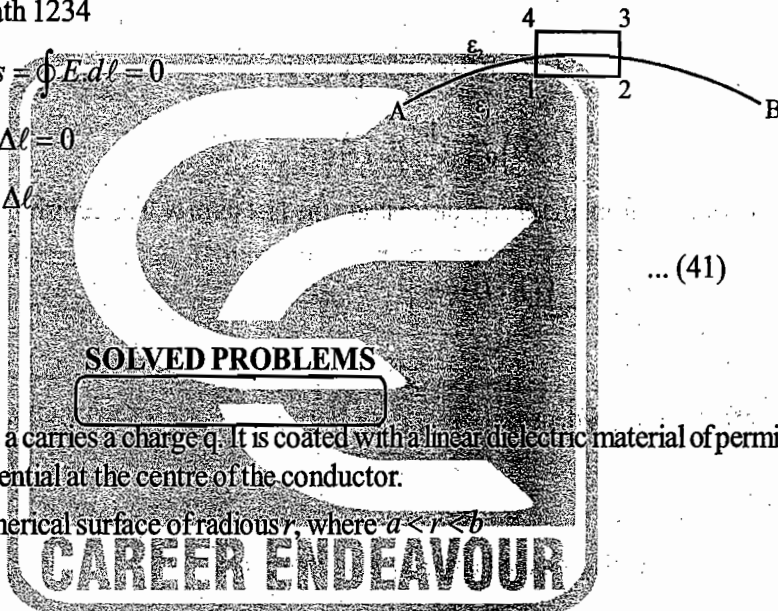
$$\text{Similarly, } E = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \quad r > b$$

Insider the metal surface the $\vec{E} = 0$

Therefore, the potential at the centre of sphere is

$$\begin{aligned} \phi &= -\int_{\infty}^a \vec{E} \cdot d\vec{r} = -\int_{\infty}^b \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} d\vec{r} - \int_b^a \frac{1}{4\pi\epsilon} \frac{q}{r^2} d\vec{r} \\ &= \frac{1}{4\pi\epsilon_0} \frac{q}{b} + \frac{q}{4\pi\epsilon} \left[\frac{1}{a} - \frac{1}{b} \right] \end{aligned}$$

This is the potential any point inside the metal sphere.



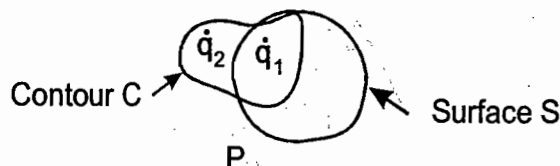


2. When a neutral dielectric is polarised, the polarization volume and surface charge appear. Show that total charge, however remain zero.

Soln. Total polarization charge

$$\begin{aligned} &= \int \rho_p dV + \int \sigma_p dS = \int -(\nabla \cdot \vec{p}) \cdot dV + \int \vec{p} \cdot \hat{n} dS \\ &= -\int \vec{p} \cdot \hat{n} dS + \int \vec{p} \cdot \hat{n} dS = 0 \quad [\text{Using divergence theorem}] \end{aligned}$$

3. Consider a set of two stationary point charges q_1 and q_2 as shown in the figure. Which of the following statements is correct? **[GATE 2002]**



- (a) The electric field at P is independent of q_2
 (b) The electric flux crossing the closed surface S is independent of q_2
 (c) The line integral of the electric field $\vec{E}$ over the closed contour C depends on q_1 and q_2 .
 (d) $\nabla \cdot \vec{E} = 0$ everywhere

Soln. (i) The electric field at p depends on both charge
 (ii) The electric flux through the closed surface S.

$$\oint \vec{E} \cdot d\vec{S} = \frac{q_1}{\epsilon_0}$$

Therefore, the electric flux crossing the closed surface S is independent of q_2 .

Correct option is (b)

Statement for Linked Answer Q. 4 and Q. 5 :

A dielectric sphere of radius R carries polarization $\vec{P} = kr^2 \hat{r}$, where r is the distance from the centre and k is a constant. In the spherical polar coordinate system, $\hat{r}$, $\hat{\theta}$ and $\hat{\phi}$ are the unit vectors.

4. The bound volume charge density inside the sphere at a distance r from the centre is :

- (a) $-4kR$ (b) $-4kr$ (c) $-4kr^2$ (d) $-4kr^3$ **[GATE 2006]**

Soln. $\rho = -\nabla \cdot \vec{P} = -\frac{1}{r^2} \frac{\partial}{\partial r} (r^2 kr^2) = -4kr$

Correct option is (b)

5. The electric field inside the sphere at a distance d from the centre is :

[GATE 2006]

- (a) $\frac{-kd^2}{\epsilon_0} \hat{r}$ (b) $\frac{-kR^2}{\epsilon_0} \hat{r}$ (c) $\frac{-kd^2}{\epsilon_0} \hat{\theta}$ (d) $\frac{-kR^2}{\epsilon_0} \hat{\theta}$

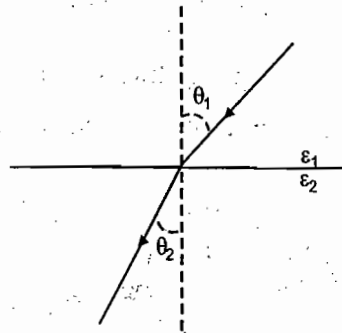
Soln. $\oint \vec{E} \cdot d\vec{S} = \int \frac{\rho \cdot dV}{\epsilon_0}$

$$\Rightarrow E \cdot 4\pi d^2 = -\int \frac{4kr}{\epsilon_0} r^2 \sin\theta d\phi d\theta dr \Rightarrow \vec{E} = -\frac{kd^2}{\epsilon_0} \hat{r}$$

Correct option is (a)



6. At the interface between two linear dielectrics (with dielectric constants (ϵ_1 and ϵ_2)), the electric field lines bend, as shown in the figure. Assume that there are no free charges at the interface. The ratio ϵ_1 and ϵ_2 is :



- (a) $\frac{\tan \theta_1}{\tan \theta_2}$ (b) $\frac{\cos \theta_1}{\cos \theta_2}$ (c) $\frac{\sin \theta_1}{\sin \theta_2}$ (d) $\frac{\cot \theta_1}{\cot \theta_2}$ [GATE 2006]

Soln. We know the boundary condition

$$D_2 \cdot \hat{n} = D_1 \cdot \hat{n}$$

$$\text{Or, } D_2 \cos \theta_2 = D_1 \cos \theta_1 \Rightarrow \epsilon_2 E_2 \cos \theta_2 = \epsilon_1 E_1 \cos \theta_1 \quad \dots (i)$$

$$\text{And } E_{1r} = E_{2r}$$

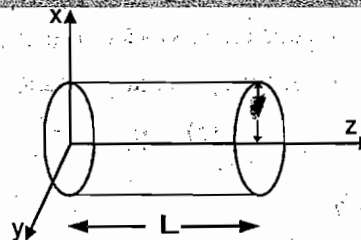
$$\Rightarrow E_2 \sin \theta_2 = E_1 \sin \theta_1 \quad \dots (ii)$$

Therefore, (ii) divided by (i), we get

$$\frac{\tan \theta_1}{\tan \theta_2} = \frac{\epsilon_1}{\epsilon_2}$$

Correct option is (a)

7. A cylindrical rod of length L and radius r , made of an inhomogeneous dielectric, is placed with its axis along the z direction with one end at the origin as shown below. [GATE 2009]



If the rod carries a polarization, $\vec{P} = (5z^2 + 7)\hat{k}$, the volume bound charge inside the dielectric is :

- (a) Zero (b) $10\pi r^2 L$ (c) $-5\pi r^2 L$ (d) $-5\pi r^2 L^2$

Soln. We have, $\vec{P} = (5z^2 + 7)\hat{k}$

Volume charge density,

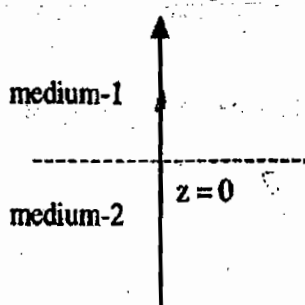
$$\rho = -\vec{\nabla} \cdot \vec{P} = -\frac{\partial}{\partial z}(5z^2 + 7) = -10z$$

Therefore, the total volume charge inside the cylinder

$$Q_{\text{total}} = -10z \cdot 2\pi r dr dz = -10 \left[\frac{z^2}{2} \right]_0^L \cdot \frac{2\pi r^2}{2} = -5\pi r^2 L^2$$



8. Two infinitely extended homogeneous isotropic dielectric media (medium-1 and medium-2 with dielectric constants $\frac{\epsilon_1}{\epsilon_0} = 2$ and $\frac{\epsilon_2}{\epsilon_0} = 5$, respectively) meet at the $z = 0$ plane as shown in the figure. A uniform electric field exists everywhere. For $z \geq 0$, the electric field is given by $\vec{E}_1 = 2\hat{i} - 3\hat{j} + 5\hat{k}$. The interface separating the two media is charge free. The electric displacement vector in the medium-2 is given by [GATE 2012]



- (a) $\vec{D}_2 = \epsilon_0 [10\hat{i} + 15\hat{j} + 10\hat{k}]$ (b) $\vec{D}_2 = \epsilon_0 [10\hat{i} - 15\hat{j} + 10\hat{k}]$
 (c) $\vec{D}_2 = \epsilon_0 [4\hat{i} - 6\hat{j} + 10\hat{k}]$ (d) $\vec{D}_2 = \epsilon_0 [4\hat{i} + 6\hat{j} + 10\hat{k}]$

Soln. Since surface is charge free

$$\therefore \vec{D}_2 \cdot \hat{n} = \vec{D}_1 \cdot \hat{n}$$

$$\therefore 5\epsilon_0 E_{2n} = 2\epsilon_0 5$$

$$E_{2n} = 2\hat{k}$$

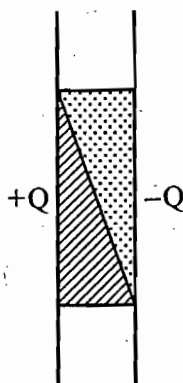
And $E_{2t} = E_{1t}$, $E_{2t} = (2\hat{i} - 3\hat{j})$

$$\therefore \vec{E}_2 = (2\hat{i} - 3\hat{j} + 2\hat{k})$$

$$\therefore \vec{D}_2 = \epsilon_0 5 (\vec{E}_2) = \epsilon_0 (10\hat{i} - 15\hat{j} + 10\hat{k})$$

Correct option is (b)

9. The space between two plates of a capacitor carrying charges $+Q$ and $-Q$ is filled with two different dielectric materials, as shown in the figure. Across the interface of the two dielectric materials, which one of the following statements is correct? [GATE 2015]



(a) $\vec{E}$ and $\vec{D}$ are continuous

(b) $\vec{E}$ is continuous and $\vec{D}$ is discontinuous

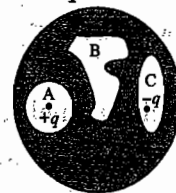
(c) $\vec{D}$ is continuous and $\vec{E}$ is discontinuous



10. A solid spherical conductor encloses 3 cavities, a cross-section of which are as shown in the figure. A net charge $+q$ resides on the outer surface of the conductor. Cavities A and C contain point charges $+q$ and $-q$, respectively. [TIFR 2015]

The net charges on the surfaces of these cavities are

- (a) $A = -q, B = q, C = 0$ (b) $A = -q, B = 0, C = -q$
 (c) $A = +q, B = 0, C = -q$ (d) $A = -q, B = 0, C = +q$



Soln. Due to external charge in the cavity the induced charge on the surface is opposite and equal magnitude of external charge.

Therefore, the charge on the surface of the cavity are $-q, 0, +q$ respectively on A, B and C cavity.





PRACTICE SET

- The electric field just outside a charged conductor having a surface density σ is:
 (a) $\frac{\sigma}{\epsilon_0}$ (b) $\frac{2\sigma}{\epsilon_0}$ (c) $\epsilon_0 \sigma$ (d) zero
- Dipolar potential varies with distance r as
 (a) $\frac{1}{r}$ (b) $\frac{1}{r^3}$ (c) $\frac{1}{r^2}$ (d) $\frac{1}{r^4}$
- In a dielectric medium if $\vec{P}$ is the polarization vector and $\vec{E}$ is the electric field then in the relation $\vec{P} = \alpha \vec{E}$, α is in general a.
 (a) scalar (b) tensor of rank 2 (c) vector (d) none of these
- Gauss's law in a dielectric medium takes the form $\oint \vec{D} \cdot d\vec{S} = q$ where q is:
 (a) total free charge enclosed (b) polarization charges
 (c) total of both free and polarization charges (d) zero
- Atomic polarizability α has the dimensions of
 (a) $C.m^{-1}.V^{-1}$ (b) $C.m^{-2}$ (c) $C.m^2.V^{-1}$ (d) none of these
- The interaction energy of an electric dipole of moment $\vec{p}$ in an external field $\vec{E}$ is
 (a) $|\vec{p} \times \vec{E}|$ (b) $\vec{p} \cdot \vec{E}$ (c) $-\vec{p} \times \vec{E}$ (d) $-\vec{p} \cdot \vec{E}$
- The net force on an electric dipole placed in a uniform electric field
 (a) depends on the dipole moment (b) depends on the strength of electric field
 (c) depends on the orientation of the dipole (d) is always zero
- The bound volume charge density in a sphere of radius a carrying polarization $\vec{p} = k\vec{r}$ is
 (a) kr (b) zero (c) k (d) $-3k$
- A point charge q is held at a distance $a/2$ from the centre of a thin conducting shell of radius a . The potential of the shell is
 (a) $\frac{1}{2\pi \epsilon_0} \cdot \frac{q}{a}$ (b) $\frac{1}{8\pi \epsilon_0} \cdot \frac{q}{a}$ (c) $\frac{1}{4\pi \epsilon_0} \cdot \frac{q}{a}$ (d) zero
- A medium in which the relations $\vec{\nabla} \cdot \vec{D} = \epsilon \vec{\nabla} \cdot \vec{E}$ and $\vec{\nabla} \cdot \vec{J} = \sigma \vec{\nabla} \cdot \vec{E}$ are true, is said to be
 (a) linear and inhomogeneous (b) linear and homogeneous
 (c) linear and isotropic (d) homogeneous and isotropic

ANSWER KEY

Questions	1	2	3	4	5
Option	(a)	(b)	(b)	(a)	(c)
Questions	6	7	8	9	10
Option	(d)	(d)	(d)	(a)	(b)

Multipole Expansion

Monopole :

A single point charge is called monopole.

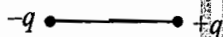
Dipole : If we put two equal and opposite charge at a small distance then it is called dipole.

Quadrupole :

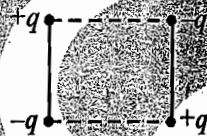
If we put two dipole of equal and opposite dipole moment of a small distance then it is called quadrupole.



(i)



(ii)



(iii)

(i) Monopole

(ii) Dipole

(iii) Quadrupole

The multipole Expansion:

Let consider a volume V and charge distribution is $\rho(r')$. The potential at a point $P(\vec{r})$ is

$$\phi(r) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(r')}{|\vec{r} - \vec{r}'|} dv$$

$$\frac{1}{|\vec{r} - \vec{r}'|} = \frac{1}{\sqrt{(\vec{r} - \vec{r}') \cdot (\vec{r} - \vec{r}')}} = \frac{1}{\sqrt{r^2 - 2\vec{r} \cdot \vec{r}' + r'^2}}$$

$$= \frac{1}{r} \left(1 + \frac{r'^2}{r^2} - \frac{2r'}{r} \cos \theta \right)^{-1/2}$$

$$= \frac{1}{r} \left[1 + \frac{r'}{r} \cos \theta + \left(\frac{r'}{r} \right)^2 \frac{1}{2} (3 \cos^2 \theta - 1) + \dots \right] \quad [\text{since } r \gg r']$$

$$\phi(\vec{r}) = \frac{1}{4\pi\epsilon_0 r} \int \rho dv + \frac{1}{4\pi\epsilon_0 r^2} \int r' \cos \theta \rho dv + \frac{1}{4\pi\epsilon_0 r^3} \int r'^2 \frac{1}{2} (3 \cos^2 \theta - 1) \rho dv + \dots$$

$$= \phi_0(r) + \phi_1(\vec{r}) + \phi_2(\vec{r}) + \dots$$

where,

$$(A) \quad \phi_0(\vec{r}) = \frac{1}{4\pi\epsilon_0 r} \int \rho dv$$

... (1)

This is known as monopole potential. It varies with distance as $\frac{1}{r}$.

$$(B) \quad \phi_1(r) = \frac{1}{4\pi\epsilon_0 r^2} \int r' \cos \theta \rho dv$$

$$= \frac{1}{4\pi\epsilon_0} \frac{1}{r^2} \int \vec{r}' \cdot \hat{r} \rho dv \quad (\text{As } r' \cos \theta = \vec{r}' \cdot \hat{r})$$

$$\phi_1(r) = \frac{1}{4\pi\epsilon_0} \left(\frac{\hat{r} \cdot \vec{p}}{r^2} \right) \text{ is called dipole term of potential.} \quad \dots (2)$$

(i) $\vec{p} = \int \vec{r}' \rho dv$ is called the dipole moment of the charge distribution. $\dots (3)$

(ii) The potential due to dipole varies with r as a function of $\frac{1}{r^2}$.

(C) Quadropole term:

$$\phi_3(\vec{r}) = \frac{1}{4\pi\epsilon_0} \frac{1}{2r^5} \int \left[3(\vec{r} \cdot \vec{r}')^2 - r^2 r'^2 \right] \rho dv$$

Let $x = x_1, y = x_2, z = x_3$

$$\therefore (\vec{r} \cdot \vec{r}')^2 = \left(\sum_{i=1}^3 x_i x_i' \right)^2 = \sum_i \sum_j x_i x_j x_i' x_j'$$

And

$$r^2 r'^2 = r'^2 \sum_i x_i'^2 = r'^2 \sum_i \sum_j x_i x_j \delta_{ij}$$

$$\phi_2(\vec{r}) = \frac{1}{4\pi\epsilon_0} \frac{1}{2r^5} \int \left(3 \sum_i \sum_j x_i x_j x_i' x_j' - \sum_i \sum_j x_i x_j \delta_{ij} r'^2 \right) \rho dv$$

$$\phi_2(r) = \frac{1}{4\pi\epsilon_0} \frac{\sum \sum x_i x_j}{2r^5} Q_{ij} \quad \dots (4)$$

(i) Potential due to quadropole moment varies with r as $\frac{1}{r^3}$.

(ii) $Q_{ij} = \int (3x_i' x_j' - \delta_{ij} r'^2) \rho dv$ is called quadropole tensor for continuous charge distribution but for point charge $Q_{ij} = \sum_k (3x_{ik} x_{jk} - \delta_{ij} r_k^2) q_k$

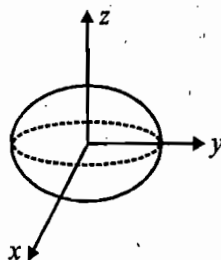
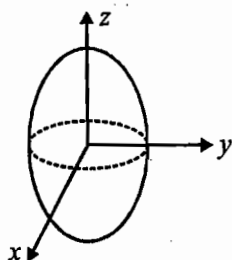
It has nine components.

(iii) For real symmetric tensor $\phi_{ij} = \phi_{ji}$ then out of nine components, six components are distinct.

(iv) If the charge distribution is symmetric about an axis, say z -axis then all the quadropole elements are expressible in terms of a single element Q_{33}

$$Q_{33} = \int (3z'^2 - r'^2) \rho dv \text{ is called quadropole moment.} \quad \dots (5)$$

(v) If $Q_{33} > 0$ then it is called prolate distribution



(vii) A spherically symmetric charge distribution ($Q_{11} = Q_{22} = Q_{33}$) has zero quadrupole moment.

Show that, if the total charge is zero then dipole moment ($\vec{p}$) is independent of the choice of origin at the coordinate system:

Suppose we consider a new coordinate system whose origin O' is at $\vec{\ell}$ with respect to the origin O of the old coordinate system. Let a point which is denoted by $\vec{r}'$ with respect to the old coordinate system and with respect to the new coordinate system.

$$\vec{r}'' = (\vec{r}' - \vec{\ell})$$

Therefore, the dipole moment with respect to the old system is

$$\begin{aligned}\vec{p} &= \int \vec{r}' \rho dv = \int (\vec{r}'' + \vec{\ell}) \rho dv \\ &= \int \vec{r}'' \rho dv + \vec{\ell} \int \rho dv = \int \vec{r}'' \rho dv + \vec{\ell} Q\end{aligned}$$

where, $Q = \int \rho dv$ = total charge

If $Q = 0$, then

$$\vec{p} = \int \vec{r}'' \rho dv \text{ dipole moment in the new system.}$$

SOLVED PROBLEMS

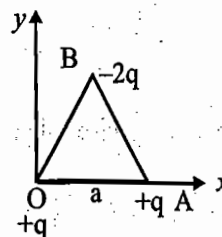
1. Suppose three point charges $+q$, $+q$, $-2q$ place at the vertices of an equilateral triangle.

Soln. Total charge, $Q = +q + q - 2q = 0$

Therefore, dipole moment is independent of choice of origin.

The dipole moment with respect to origin.

$$\begin{aligned}&= q \times 0 + q(a\hat{i}) - 2q\left(\frac{a}{2}\hat{i} + \frac{\sqrt{3}}{2}\hat{j}\right) \\ &= 0 + qa\hat{i} - qa\hat{i} - q\sqrt{3}a\hat{j} = -\hat{j}\sqrt{3}aq = -\hat{j}\sqrt{3}aq\end{aligned}$$



2. A sphere with a charge distribution $\sigma(\theta) = \sigma_0 \cos \theta$. Find the dipole moment of the sphere.

Soln. Let elementary area,

$$\ell_s = r^2 \sin \theta' d\theta' d\phi'$$

$$\text{And } \sigma(\theta_0) = \sigma_0 \cos \theta'$$

$$\therefore dq = \sigma_0 \cos \theta' r^2 \sin \theta' d\theta' d\phi' = 0$$

So, the dipole moment is independent of choice of origin.

Let any point on the surface element given by

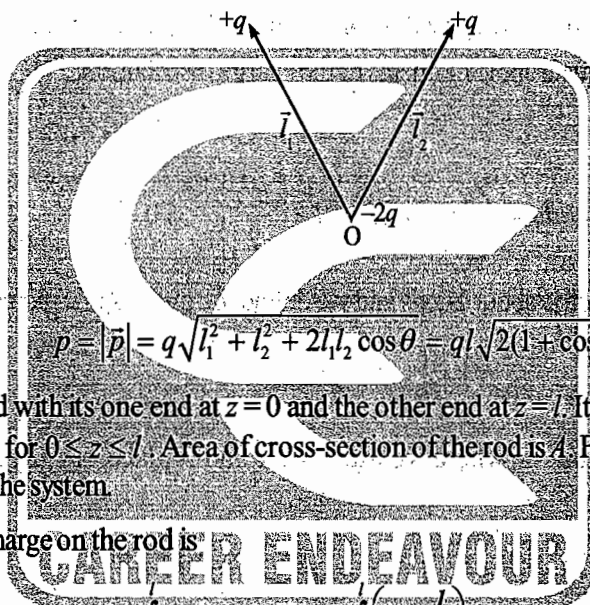
Therefore, the dipole moment,

$$\begin{aligned}\vec{p} &= \int \vec{r}' dq = r^3 \sigma_0 \int_0^\pi \int_0^{2\pi} \left[(\sin^2 \theta' \cos \theta' \cos \phi') \hat{i} + (\sin^2 \theta' \cos \theta' \sin \phi') \hat{j} + (\cos \theta' \sin \theta') \hat{k} \right] d\phi' d\theta' \\ &= 0 + 0 + r^3 \sigma_0 \int_0^\pi \cos^2 \theta' \sin \theta' d\theta' \int_0^{2\pi} d\phi' \hat{k} = \frac{4}{3} \pi r^3 \sigma_0 \hat{k}\end{aligned}$$

3. Consider a system of three charges: $-2q$ charge at the origin, $+q$ at $\vec{l}_1$ and another charge $+q$ at $\vec{l}_2$. Find the dipole moment $\vec{p}$ of the system. What is the magnitude of $\vec{p}$ if $|\vec{l}_1| = |\vec{l}_2| = l$ and angle between $\vec{l}_1$ and $\vec{l}_2$ is θ .

Soln. As the total charge is zero the dipole moment will be independent of the choice of origin. Now the dipole moment with respect to the origin is

$$\vec{p} = \int \vec{r}' dq = \vec{l}_1 \cdot (+q) + \vec{l}_2 \cdot (+q) + 0 \cdot (-2q) = q(\vec{l}_1 + \vec{l}_2).$$



Its magnitude

$$p = |\vec{p}| = q \sqrt{l_1^2 + l_2^2 + 2l_1 l_2 \cos \theta} = ql \sqrt{2(1 + \cos \theta)}.$$

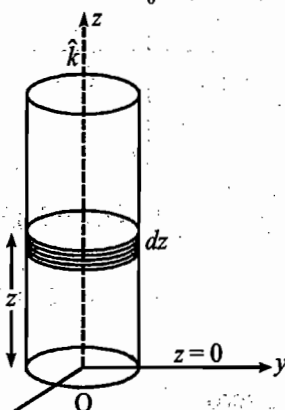
4. Consider a thin rod with its one end at $z=0$ and the other end at $z=l$. Its charge density is given by $\rho(z) = \alpha(z - l/2)$ for $0 \leq z \leq l$. Area of cross-section of the rod is A . Find the total charge on the rod and dipole moment of the system.

Solution : Total charge on the rod is

$$\int_0^l \rho(z) \cdot A dz = \alpha A \int_0^l \left(z - \frac{l}{2} \right) dz = 0.$$

So the dipole moment is independent on the choice of the origin. Now the dipole moment with respect the origin O (Fig. 3.P-2) is

$$\vec{p} = \int \vec{r}' dq = \hat{k} \int_0^l z \cdot (\rho A dz) = \hat{k} \alpha A \int_0^l z \left(z - \frac{l}{2} \right) dz = \hat{k} \alpha A \cdot \frac{l^3}{12}.$$

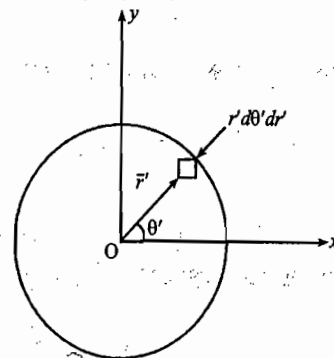




5. Consider a circular disc of radius a in the xy -plane centered on the origin and containing a surface charge density $\sigma(r, \theta) = \sigma_0 r \cos \theta$, where θ is the angle made by the position vector with a fixed diameter chosen as x -axis. Find the dipole moment of the charge distribution.

Soln. Considering a surface element $r'd\theta' \cdot dr'$ about the point $\vec{r}'$ we can write for the dipole moment about the origin as

$$\begin{aligned}\vec{p} &= \int \vec{r}' dq = \int (x'\hat{i} + y'\hat{j}) \sigma \cdot r' d\theta' dr' \\ &= \int (r' \cos \theta' \hat{i} + r' \sin \theta' \hat{j}) \sigma_0 \cdot r' \cos \theta' \cdot r' d\theta' dr' \\ &= \hat{i} \sigma_0 \int_0^a r'^3 dr' \int_0^{2\pi} \cos^2 \theta' d\theta' + \hat{j} \sigma_0 \int_0^a r'^3 dr' \int_0^{2\pi} \sin \theta' \cos \theta' d\theta' \\ &= \hat{i} \sigma_0 \frac{a^4}{4} \cdot \pi + 0 = \hat{i} \sigma_0 \pi a^4 / 4.\end{aligned}$$



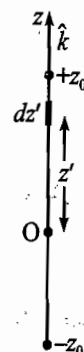
6. Consider a uniform line distribution of charge extending from $z = -z_0$ to $z = z_0$ and having a charge λ per unit length. Find the dipole and quadrupole moments of the charge distribution.

Solution : Dipole moment

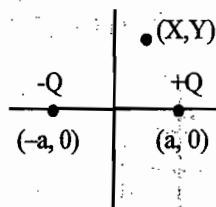
$$\vec{p} = \int \vec{r}' dq = \int_{-z_0}^{+z_0} kz' (\lambda dz') = k\lambda \left[\frac{z'^2}{2} \right]_{-z_0}^{+z_0} = 0.$$

The charge distribution is symmetric about z -axis and we define the quadrupole moment,

$$\begin{aligned}Q_{33} &= \int_V (3z'^2 - r'^2) \rho dV \\ &= \int_{-z_0}^{+z_0} (3z'^2 - z'^2) \lambda dz' \quad [\text{where we use } \vec{r}' = z'\hat{k} \text{ and replace } \rho dV \text{ by } \lambda dz'] \\ &= 2\lambda \cdot \left[\frac{z'^3}{3} \right]_{-z_0}^{+z_0} = \frac{4}{3} \lambda z_0^3.\end{aligned}$$



7. Consider two charges $+Q$ and $-Q$ placed at the points $(a, 0)$ and $(-a, 0)$ in a plane, as shown in the figure on the right. If the origin is moved to the point (X, Y) , the magnitude of the dipole moment of the given charge distribution with respect to this origin will be [TIFR 2013]



- (a) $Q\sqrt{(a-X)^2 + Y^2} - Q\sqrt{(a+X)^2 + Y^2}$ (b) $Q(a-X) - Q(-a+X)$
(c) $2Qa$ (d) $2Q\sqrt{X^2 + Y^2}$

Soln. Total charge, $Q_T = -Q + Q = 0$

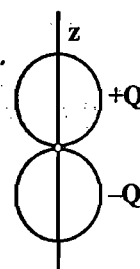
Since the total charge of the system is zero.

Therefore, the dipole moment does not depend on origin.

Distance to the origin of old coordinate $= Q \cdot 2a =$ the dipole moment with



8. An electric dipole is constructed by fixing two circular charged rings, each of radius 'a', with an insulating contact (see figure). One of these rings has total charge +Q and the other has total charge -Q. If the charge is distributed uniformly along each ring, the dipole moment about the point of contact will be



- (a) $\frac{Qa}{\pi} \hat{z}$ (b) $4\pi Qa \hat{z}$
(c) $2Qa \hat{z}$ (d) Zero

[TIFR 2014]

Soln. Since charge is uniformly distributed over the surface of ring so we can treat them as a point charge at the center of the ring.

The dipole moment with respect to the contact point is

$$P = Qa\hat{z} + (-a)(-Q)\hat{z} = 2aQ\hat{z}$$

9. Consider a two dimensional quadrupole consisting of four charges +q, -q, +q and -q located at the points $(a\sqrt{2}, 0)$, $(0, a\sqrt{2})$, $(-a\sqrt{2}, 0)$ and $(0, -a\sqrt{2})$, respectively. Show that the quadrupole moment tensor takes the form

$$\begin{pmatrix} 12qa^2 & 0 & 0 \\ 0 & -12qa^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

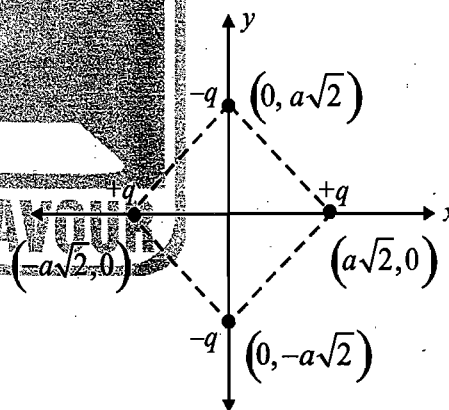
Soln. Coordinates of charges are

$$(x_1, y_1, z_1) = (a\sqrt{2}, 0, 0)$$

$$(x_2, y_2, z_2) = (0, a\sqrt{2}, 0)$$

$$(x_3, y_3, z_3) = (-a\sqrt{2}, 0, 0)$$

$$(x_4, y_4, z_4) = (0, -a\sqrt{2}, 0)$$



The quadrupole due point charge distribution

$$Q_{ij} = \sum_k (3x_{ik}x_{jk} - \delta_{ij}r_k^2) q_k$$

$$Q_{11} = \sum_{k=1}^4 (3x_k^2 - r_k^2) q_k = \sum_{k=1}^4 (2x_k^2 - y_k^2 - z_k^2) q_k$$

$$= (2 \cdot 2a^2 - 0 - 0)q - q(0 - 2a^2 - 0) + q(2 \cdot 2a^2 - 0 - 0) - q(0 - 2a^2 - 0)$$

$$= 4a^2q + 2a^2q + 4a^2q + 2a^2q = 12a^2q$$

$$Q_{22} = \sum_{k=1}^4 (3y_k^2 - r_k^2) q_k = \sum_{k=1}^4 (2y_k^2 - x_k^2 - z_k^2) q_k$$

$$= (0 - 2a^2 - 0)q - q(4a^2 - 0 - 0) + q(0 - 2a^2 - 0) - q(4a^2 - 0 - 0) = -12a^2q$$



$$= -2a^2q - q(-2a^2) + q(-2a^2) - q(-2a^2) = 0$$

$$Q_{12} = \sum_{k=1}^4 x_k y_k q_k = a\sqrt{2} \cdot 0 \cdot q - 0 \cdot a\sqrt{2}q + q \cdot (-a\sqrt{2}) \cdot 0 - q \cdot (-a\sqrt{2}) \cdot 0 = 0$$

$$Q_{21} = 0$$

$$Q_{13} = \sum_{k=1}^4 x_k z_k q_k = 0 = Q_{31}$$

$$Q_{23} = \sum_{k=1}^4 y_k z_k q_k = 0 = Q_{32}$$

Therefore, the quadrupole tensor is

$$Q_{ij} = \begin{pmatrix} 12a^2q & 0 & 0 \\ 0 & -12a^2q & 0 \\ 0 & 0 & 0 \end{pmatrix}$$





PRACTICE SET

1. The electric field due to an electric quadrupole varies with distance r as
 (a) $\frac{1}{r^2}$ (b) $\frac{1}{r^4}$ (c) $\frac{1}{r^3}$ (d) none of these
2. If the total charge of a system is zero then its dipole moment is
 (a) zero (b) independent on the choice of origin
 (c) dependent on the choice of origin (d) none of these
3. Four charges $q, q, -2q$ and q are placed at the corners of a square of side a having coordinates $(0, 0), (a, 0), (a, a)$ and $(0, a)$, respectively. The electric dipole moment of the configuration about the origin is
 (a) $qa(\hat{i} + \hat{j})$ (b) $qa(\hat{i} - \hat{j})$ (c) $-qa(\hat{i} + \hat{j})$ (d) $-qa(\hat{i} - \hat{j})$
4. The electric field due to an electric dipole varies with distance r as
 (a) $\frac{1}{r^2}$ (b) $\frac{1}{r^4}$ (c) $\frac{1}{r^3}$ (d) none of these
5. For a spherically symmetric charge distribution quadrupole moment is
 (a) always positive (b) always zero
 (c) always negative (d) undefined

ANSWER KEY					
Questions	1	2	3	4	5
Option	(b)	(b)	(c)	(c)	(b)

CAREER ENDEAVOUR

Capacitor

Introduction:

A capacitor is a device which stores electric charge. Capacitors vary in shape and size, but the basic configuration is two conductors carrying equal but opposite charges. Capacitors have many important applications in electronics. Some examples resistors, filtering out unwanted frequency signals, forming resonant circuits and making frequency-dependent and independent voltage dividers when combined with resistors.

Capacitors:

In uncharged state, the charge on the both conductor in the capacitor is zero. During the charging process a charge Q is moved from one conductor to the other one giving one conductor a charge $+Q$ and the other one a charge $-Q$. A potential difference ΔV is created, with positively charged conductor at higher potential than the negatively charged conductor. Note that whether charged and uncharged, the net charge on the capacitor as a whole is zero.

Let the amount of charged Q stored in the capacitor and it produce voltage difference ΔV . Thus we may write,

$$Q \propto \Delta V$$

Or,

$$Q = C|\Delta V| \quad \Rightarrow \quad C = \frac{Q}{|\Delta V|} \quad (1)$$

where, C is a positive constant called capacitance.

The S.I. unit of capacitor is the Farad (F).

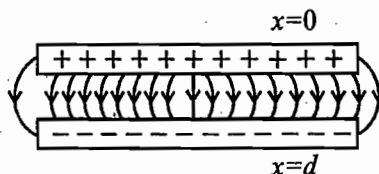
1F = 1 farad = 1 coulomb/volt

$$1\text{F} = 10^{+12} \text{PF} \quad 1\text{F} = 10^6 \mu\text{F}$$

Calculation of Capacitance:

1. Parallel-plate capacitor:

Consider two metallic plates of equal area A separated by a distance d , as shown in figure. The top plate carries a charge $+Q$. While the bottom plate carries a charge $-Q$.



From Gaussian law we can say that the field exist only inside the plate and it is given by

$$\vec{E} = \frac{\sigma}{\epsilon_0}$$

$$V = \int_d^0 E \cdot d\ell = + \int_0^d E \cdot d\ell = \frac{\sigma}{\epsilon_0} d$$

Therefore, the capacitance,

$$C = \frac{Q}{V} = \frac{\sigma A}{V} = \frac{\sigma A \epsilon_0}{\sigma d} = \frac{A \epsilon_0}{d}$$

Or,

$$C = \frac{A \epsilon_0}{d}$$

... (2)

Special Cases:

Capacitor with a composite dielectric medium:

Let a capacitor contain two linear dielectric of thickness t_1 and t_2 and permittivities ϵ_1 and ϵ_2 respectively.

The field inside the two medium is

$$E_1 = \frac{\sigma}{\epsilon_1} \text{ and } E_2 = \frac{\sigma}{\epsilon_2}$$

Therefore, potential difference between the plates would be

$$V = - \int_{t_1+t_2}^0 E \cdot dx = - \int_{t_1+t_2}^{t_1} E_2 dx - \int_{t_1}^0 E_1 dx$$

$$= - \frac{\sigma}{\epsilon_2} [t_1 + t_2 - t_1] - \frac{\sigma}{\epsilon_1} t_1 = - \frac{\sigma}{\epsilon_2} t_2 - \frac{\sigma}{\epsilon_1} t_1$$

$$C = \frac{Q}{V} = \frac{\sigma A}{\sigma \left(\frac{t_1}{\epsilon_1} + \frac{t_2}{\epsilon_2} \right)}$$

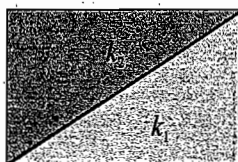
$$C = \frac{\epsilon_0 A}{\frac{t_1}{k_1} + \frac{t_2}{k_2}}$$

Therefore, the capacitor with several dielectric medium

$$C = \frac{\epsilon_0 \cdot A}{\sum_i \frac{t_i}{k_i}}$$

... (3)

1. Find the capacitance of a parallel plate capacitor whose upper triangle and lower triangle have dielectric constant k_1 and k_2 respectively as shown in figure.



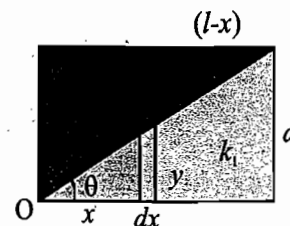
Let consider element at x distance from left side.

Therefore, the capacitance of the element is

$$dC = \frac{\epsilon_0 b dx}{\left(\frac{y}{k_1} + \frac{y-d}{k_2} \right)}$$



$$\begin{aligned}
 &= \varepsilon_0 b \int_0^\ell \frac{dx}{\frac{x \tan \theta}{k_1} + \frac{(\ell-x) \tan \theta}{k_2}} = \frac{\varepsilon_0 b}{\tan \theta} \int_0^\ell \frac{dx}{\frac{x}{k_1} + \frac{(\ell-x)}{k_2}} \\
 &= \frac{\varepsilon_0 b k_1 k_2}{\tan \theta (k_2 - k_1)} \left[\ln(k_2 - k_1)x + \ell k_1 \right]_0^\ell \\
 &= \frac{\varepsilon_0 b \ell}{d} \left(\frac{k_1 k_2}{k_2 - k_1} \right) \ln \left(\frac{k_2 \ell}{\ell k_1} \right) = \frac{\varepsilon_0 A}{d} \frac{k_1 k_2}{k_2 - k_1} \ln \frac{k_2}{k_1}
 \end{aligned}$$



2. Capacitor with a non-uniform dielectric:

Let the dielectric constant at P at a distance x from the plate 1 be k and α .

$$\frac{dk}{dx} = \alpha$$

$$dk = \alpha dx$$

$$k = \alpha x + \beta$$

Now, at $x=0$, $k=k_1$ and at $x=d$, $k=k_2$

$$\therefore k = \frac{(k_2 - k_1)x}{d} + k_1$$

Now the field at P is

$$E = \frac{\sigma}{\varepsilon} = \frac{\sigma}{\varepsilon_0 k} = \frac{\sigma}{\varepsilon_0 \left[\frac{(k_2 - k_1)x}{d} + k_1 \right]}$$

$$\therefore V = - \int_d^0 E dx = \frac{\sigma}{\varepsilon_0} \int_0^d \frac{dx}{\frac{(k_2 - k_1)x}{d} + k_1}$$

$$= \frac{\sigma d}{\varepsilon_0 (k_2 - k_1)} \ln \left(\frac{k_2 - k_1}{d} x + k_1 \right) \Big|_0^d$$

$$= \frac{\sigma d}{\varepsilon_0 (k_2 - k_1)} \ln \left(\frac{k_2 - k_1 + k_1}{k_1} \right) = \frac{Qd}{\varepsilon_0 A (k_2 - k_1)} \ln \left(\frac{k_2}{k_1} \right)$$

$$\therefore c = \frac{Q}{V} = \frac{\varepsilon_0 A (k_2 - k_1)}{d \ln(k_2 / k_1)} \quad \dots (4)$$

(2) Cylindrical capacitors:

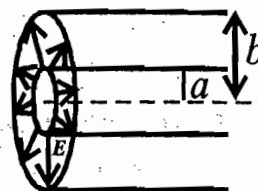
Consider next a solid cylindrical conductor of radius a surrounded by a coaxial cylindrical shell of inner radius b as shown in figure. The length of both cylinders so that edge effects can be neglected. The capacitor is charged so that the inner cylinder has charge $+Q$. While the out shell has a charge $-Q$.

$$\oint E \cdot dA = \frac{\lambda l}{\epsilon_0}$$

$$\Rightarrow E \cdot 2\pi r l = \frac{\lambda l}{\epsilon_0}$$

$$\Rightarrow \vec{E} = \frac{\lambda}{2\pi r \epsilon_0} \hat{r} \quad a < r < b$$

$$= 0 \quad r > b, r < a$$



Therefore, the potential difference,

$$V = - \int_b^a E \cdot dr = - \frac{\lambda}{2\pi \epsilon_0} \int_a^b \frac{dr}{r} = - \frac{\lambda}{2\pi \epsilon_0} \ln \left(\frac{b}{a} \right)$$

Therefore, the capacitance, $c = \frac{q}{V} = \frac{\lambda \times 2\pi \epsilon_0 \ln(b/a)}{\lambda \ln(b/a)} = \frac{2\pi \epsilon_0 l}{\ln(b/a)}$

$$c = \frac{2\pi \epsilon_0 L}{\ln(b/a)}$$

... (5)

Therefore, capacitor per unit length

$$c = \frac{2\pi \epsilon_0}{\ln(b/a)}$$

... (6)

(iii) Spherical capacitor

Let consider a spherical capacitor consist of pair of concentric metal spheres, the space between them being filled with a dielectric and a and b the radius of smaller and larger sphere, the charge of outer is sphere is $-Q$ and inner sphere $+Q$.

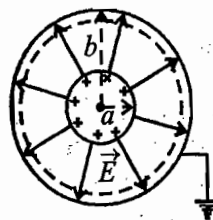
Therefore, the field,

$$\oint \vec{E} \cdot d\vec{S} = \frac{Q}{\epsilon_0}$$

$$\Rightarrow E \cdot 4\pi r^2 = \frac{Q}{\epsilon_0}$$

$$\Rightarrow \vec{E} = \frac{1}{4\pi \epsilon} \frac{Q}{r^2} \quad a < r < b$$

$$= 0 \quad r > b, r < a$$



Therefore, the potential difference is

$$V = \int_b^a E \cdot dn = \frac{1}{4\pi \epsilon_0} Q \left(\frac{1}{a} - \frac{1}{b} \right)$$

Therefore, the capacitor



$$C = \frac{Q}{V}$$

$$C = \frac{4\pi\epsilon_0 ab}{(b-a)}$$

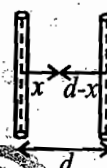
... (7-a)

(iv) Capacitor between two long thin parallel wires in air:

Let two wire carries charge $+\lambda$ and $-\lambda$ per unit length and their centre to centre distance d .
Let the field at any point (P) inside them is

$$E = \left[\frac{\lambda}{2\pi\epsilon_0 x} + \frac{\lambda}{2\pi\epsilon_0 (d-x)} \right] \hat{x}$$

$$\therefore V = - \int_{d-a}^a E \cdot dx = \frac{\lambda}{2\pi\epsilon_0} [\ln x - \ln(d-x)]_a^{d-a}$$



$$= \frac{\lambda}{\pi\epsilon_0} \ln \left(\frac{d-a}{a} \right)$$

Therefore, the capacitance, $C = \frac{Q}{V} = \frac{\lambda L}{\frac{\lambda}{\pi\epsilon_0} \ln \left(\frac{d-a}{a} \right)}$

$$C = \frac{L\pi\epsilon_0}{\ln \left(\frac{d-a}{a} \right)}$$

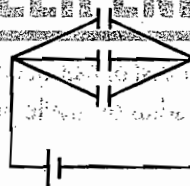
... (7-b)

Combination of capacitor:

(1) Parallel combination:

$$C_T = c_1 + c_2 + c_3 \dots$$

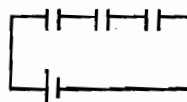
$$C_T = \sum_i c_i$$



... (8)

(ii) Series combination:

$$\frac{1}{C_T} = \frac{1}{c_1} + \frac{1}{c_2} + \frac{1}{c_3}$$



$$\Rightarrow \frac{1}{C_T} = \sum_i \frac{1}{c_i}$$

... (9)

Energy stored in a charged capacitor :

Suppose at any intermediate stage during the capacitor charging the charge on the positive conducto
is q and potential difference across the capacitor is $\phi = \frac{q}{C}$ work done to increase the charge by an amount d



$$W = \int_0^Q \frac{q}{c} dq = \frac{Q^2}{2c}$$

$$W = \frac{Q^2}{2c}$$

... (10)

Also, $Q = cV$

$$W = \frac{1}{2} cV^2$$

... (11)

SOLVED PROBLEMS

1. The conductors in a 0.75 km long two-wire transmission line are separated by a centre-to-centre distance of 0.2m. If each conductor has a diameter of 4cm, then the capacitance of the line is

[JEST 2013]

- (a) 8.85 μF (b) 88.5 nF (c) 8.85 pF (d) 8.85 nF

Soln. We capacitance of two line wire is

$$C = \frac{L\pi\epsilon_0}{\ln\left(\frac{d-a}{a}\right)} = \frac{0.75 \times 10^3 \times 3.14 \times 8.85 \times 10^{-12}}{\ln\left(\frac{0.2-0.02}{0.02}\right)} = \frac{20.84 \times 10^{-9}}{\ln 9}$$

$$= 9.48 \times 10^{-9} \text{ F} = 9.48 \text{ nF}$$

Correct option is (d)

2. A coaxial cable of uniform cross-section contains an insulating material of dielectric constant 3.5. The radius of the central wire is 0.01 m and that of the sheath is 0.02 m. The capacitance per kilometer of a cable is :

[GATE 2001]

- (a) 280.5 nF (b) 28.05 nF (c) 56.10 nF (d) 2.805 nF

Soln. Capacitance of coaxial cable is per unit length

$$C = \frac{2\pi\epsilon}{\ln(b/a)} = \frac{2\pi 3.5 \times 8.85 \times 10^{-12}}{\ln\left(\frac{0.02}{0.01}\right)} = 280.63 \times 10^{-12} \text{ F/m} = 280.63 \text{ nF/km}$$

Correct option is (a)

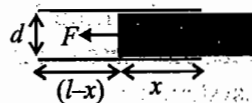
3. Consider a parallel plate capacitor with plate separation d and each plate having a length l and width a . A dielectric slab of permittivity ϵ , thickness d , length l and width a is partially inserted between the plates. If a constant pd V is maintained between the plates, find the force exerted on the dielectric slab when a length x of the slab remains between the plates.

Soln. The system may be considered as the parallel combination of two capacitors C_1 and C_2 , where,

$$C_1 = \frac{\epsilon ax}{d} \text{ and } C_2 = \frac{\epsilon_0 a(l-x)}{d}$$

The total capacitance is then

$$C = C_1 + C_2 = \frac{a}{d} [(\epsilon - \epsilon_0)x + \epsilon_0 l]$$





$$U = \frac{1}{2} CV^2 = \frac{a}{2d} [(\epsilon - \epsilon_0)x + \epsilon_0 \ell] V^2$$

As V is constant the force experienced by the slab will be

$$\vec{F} = \frac{\partial U}{\partial x} \bigg|_V \hat{x} = \frac{a}{2d} (\epsilon - \epsilon_0) V^2 \hat{x}$$

where $\hat{x}$ is a unit vector in the direction of increasing x . This means that the force is attractive.

4. Consider a spherical charge distribution with a volume density

$$\rho(r) = \begin{cases} \rho_0 \cdot \frac{a}{r} & \text{for } 0 < r < a \\ 0 & \text{for } r > a \end{cases}$$

Show that the total electrostatic energy of the system is $\frac{Q^2}{6\pi\epsilon_0 a}$, where Q is the total charge in the system.

Soln. Total charge in the system is

$$Q = \int_0^a \rho(r) \cdot 4\pi r^2 dr = 4\pi\rho_0 a \int_0^a r dr = 2\pi\rho_0 a^3$$

The electric field at any outside point can be obtained from Gauss's law as

$$\int \vec{E}_0 \cdot d\vec{S} = \frac{Q}{\epsilon_0} \text{ or } E_0 \cdot 4\pi r^2 = \frac{Q}{\epsilon_0} \Rightarrow E_0 = \frac{Q}{4\pi\epsilon_0 r^2}$$

For an internal point,

$$\int \vec{E}_i \cdot d\vec{S} = \frac{Q_i}{\epsilon_0} = \frac{1}{\epsilon_0} \int_0^r \rho(r) \cdot 4\pi r^2 dr = \frac{2\pi\rho_0 a r^2}{\epsilon_0}$$

$$\therefore E_i \cdot 4\pi r^2 = \frac{Q r^2}{\epsilon_0 a^2} \text{ or } E_i = \frac{Q}{4\pi\epsilon_0 a^2}$$

Now, the total energy in the system is

$$\begin{aligned} U &= \int_0^a \frac{1}{2} \epsilon_0 E_i^2 \cdot 4\pi r^2 dr + \int_a^\infty \frac{1}{2} \epsilon_0 E_0^2 \cdot 4\pi r^2 dr \\ &= 2\pi\epsilon_0 \int_0^a \left(\frac{Q}{4\pi\epsilon_0 a^2} \right)^2 r^2 dr + 2\pi\epsilon_0 \int_a^\infty \left(\frac{Q}{4\pi\epsilon_0 r^2} \right)^2 \cdot \frac{dr}{r^2} = \frac{Q^2}{6\pi\epsilon_0 a} \end{aligned}$$

5. A parallel plate air capacitor has the plates of area A each.

(a) assuming that the charge Q on it is kept constant find the work done against the electrical forces to increase the plate separation from d_1 to d_2 .

(b) Assuming that the $p.d$ V across it is maintained constant find the change in energy of the system.

Soln. The energy of the system is

$$W = \Delta U = \frac{Q^2}{2\epsilon_0 A} (d_2 - d_1)$$

(b) When the pd is maintained constant the change in energy of the system would be

$$\Delta U = \frac{\epsilon_0 AV^2}{2} \left(\frac{1}{d_2} - \frac{1}{d_1} \right)$$

This is negative. To understand it we must consider the work done against the electrical forces as well as the energy supplied by the source, which maintained the pd constant.

6. Consider a capacitor consisting of two fixed semicircular plates of radius R separated by a distance d . A semicircular dielectric slab of dielectric constant K almost fills up the space between the plates. The slab is free to rotate about an axis normal to the plates and passing through the centre. A constant pd V is maintained between the plates. If the slab is now rotated through an angle θ show that the torque tending

to pull the slab back inside the capacitor would be $\frac{(K-1)\epsilon_0 R^2 V^2}{4d}$

Soln. The system may be considered as the parallel combination of two capacitors C_1 and C_2 , where $\tilde{C}_1$ and C_2 are the capacitances of the parts with and without the dielectric. Thus, capacitance

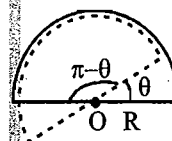
$$C = C_1 + C_2 = \frac{\epsilon \cdot A_1}{d} + \frac{\epsilon_0 \cdot A_2}{d} = \frac{\epsilon}{d} \cdot \frac{1}{2} R^2 (\pi - \theta) + \frac{\epsilon_0}{d} \cdot \frac{1}{2} R^2 \theta$$

The electrostatic energy of the system is

$$U = \frac{1}{2} CV^2$$

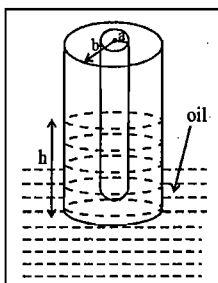
As the $p d V$ is constant the torque acting on the system is

$$\tau = \frac{\partial U}{\partial \theta} = \frac{V^2}{2} \cdot \frac{\partial C}{\partial \theta} = \frac{V^2}{2} \cdot \frac{R^2}{2d} (\epsilon_0 - \epsilon) = -\frac{(K-1)\epsilon_0}{2} R^2 V^2 / 4d$$



The negative sign indicates that the torque tries to decrease the angle θ .

7. Two coaxial cylindrical metal tubes (inner radius 'a' and outer radius 'b') stand vertically in a tank of dielectric oil. The susceptibility of the oil is χ_e and mass density is ρ . The inner tube is maintained at potential V , and the outer one is grounded. To what height (h) does the oil rise in the space between the tubes?



Soln. Let any time t the liquid rise up certain height x .

The capacitance of empty region

$$c_1 = \frac{2\pi\epsilon_0(L-x)}{\ln(b/a)}$$

The capacitance of filled with oil region



Since these two capacitors connected with same potential i.e. they are in parallel
Therefore, the total capacitance,

$$C = C_1 + C_2 = \frac{2\pi\epsilon_0}{\ln(b/a)} \{(L-x) + (1+\chi)x\}$$

$$\therefore F = -\frac{dU}{dx} = -\frac{1}{2}V^2 \frac{dC}{dx} = \frac{1}{2}V^2 \frac{2\pi\epsilon_0}{\ln(b/a)} (1-\chi)$$

$$\therefore \frac{1}{2}V^2 \frac{2\pi\epsilon_0}{\ln(b/a)} (1-\chi) = \pi(b^2 - a^2) \rho g h$$

$$\Rightarrow h = \frac{V^2 \epsilon_0 (1-\chi)}{\ln(b/a) (b^2 - a^2) \rho g}$$





PRACTICE SET

- If the separation between two charges is increased the electric potential energy
 - always decreases
 - may increase or decrease
 - always increases
 - remains the same.
- Energy density of the electric field produced by a point charge varies with the distance r from the point charge as:
 - $\frac{1}{r^2}$
 - $\frac{1}{r^4}$
 - $\frac{1}{r^3}$
 - $\frac{1}{r^5}$
- If two capacitors having the same capacitance C and breakdown voltage V are connected in series, their equivalent capacitance and the breakdown voltage would be
 - $\frac{C}{2}$ and $\frac{V}{2}$
 - $2C$ and $2V$
 - $\frac{C}{2}$ and $2V$
 - $2C$ and V
- If two capacitors having the same capacitance C and breakdown voltage V are connected in parallel, their equivalent capacitance and the breakdown voltage would be
 - $\frac{C}{2}$ and $\frac{V}{2}$
 - $2C$ and $2V$
 - $\frac{C}{2}$ and $2V$
 - $2C$ and V
- A dielectric slab is introduced between the plates of an isolated charged parallel plate air capacitor. Which of the following quantities will remain unchanged?
 - charge on the capacitor
 - energy of the capacitor
 - pd across the capacitor
 - electric field inside the capacitor
- Capacitance has the dimensions
 - $[M^{-1}L^{-2}T^4I^2]$
 - $[MLT^{-3}I^{-1}]$
 - $[MLT^{-4}I^{-2}]$
 - $[M^{-1}L^{-1}T^3I]$
- A fixed pd V is applied between the plates of a parallel plate capacitor. Now a conducting plate is introduced between the plates such that it divides the space between positive and negative plates in the ratio $d_1 : d_2$. Then the voltage of the conducting plate with respect to the negative plate would be
 - zero
 - $\frac{d_2}{d_1 + d_2}V$
 - V
 - $\frac{d_1}{d_1 + d_2}V$
- The pd across a capacitor is kept constant. If a dielectric slab of dielectric constant K is introduced between the plates, the stored energy will
 - decrease by a factor K
 - increase by a factor K
 - remain constant
 - increase or decrease depending on the nature of the dielectric
- A parallel plate capacitor of plate area A and plate separation d is charged to a pd of V_0 . It is now disconnected from source of pd and then its plate separation is increased to $3d$. Workd necessary for this is
 - $3\epsilon_0 AV_0^2/d$
 - $\epsilon_0 AV_0^2/3d$
 - $\epsilon_0 AV_0^2/2d$
 - $\epsilon_0 AV_0^2/d$
- A metal sphere of radius r_1 carrying a charge q_1 is connected by a thin wire to another metal sphere of radius r_2 carrying a charge q_2 . The common potential of the system is
 - $\frac{1}{4\pi\epsilon_0} \cdot \frac{q_1 + q_2}{r_1 + r_2}$
 - $\frac{1}{4\pi\epsilon_0} \cdot \left(\frac{q_1}{r_1} - \frac{q_2}{r_2} \right)$
 - $\frac{1}{4\pi\epsilon_0} \cdot \left(\frac{q_1}{r_1} + \frac{q_2}{r_2} \right)$
 - $\frac{1}{4\pi\epsilon_0} \cdot \frac{q_1 q_2}{\sqrt{r_1 r_2}}$

ANSWER KEY

Questions	1	2	3	4	5
Option	(b)	(b)	(c)	(d)	(a)

Chapter 6

Laplace equation & Boundary value problem

Introduction:

The primary problem in electrostatics is to find the electric field due to a given stationary charge distribution. If the charge distribution is specified everywhere then in principle the electric field can be found by using Coulomb's law. However, there is a large class of problems in which we do not know the charge distribution and consequently we cannot directly determine E and ϕ . Specially where $\rho = 0$ we need to solve Laplace's equation with appropriate boundary conditions.

Poisson's equation and Laplace equation:

We have found in chapter -1 that the divergence of electric field is charge density divided by ϵ_0

$$\nabla \cdot \vec{E} = \rho / \epsilon_0$$

And we defined potential V as $\vec{E} = -\nabla V$

$$\nabla \cdot \vec{E} = \nabla \cdot (-\nabla V) = -\nabla \cdot (\nabla V) = -\nabla^2 V$$

$$\text{Hence, } -\nabla^2 V = \frac{\rho}{\epsilon_0}$$

$$\Rightarrow \nabla^2 V = -\frac{\rho}{\epsilon_0} \quad (1)$$

This is Poisson's equation when there is no charge ($\rho = 0$) this reduced to Laplace equation

$$\Rightarrow \boxed{\nabla^2 V = 0} \quad \dots(2)$$

Laplace equation in different dimension and coordinate systems

The vector equation $\nabla^2 V = 0$ reduces to

$$\text{1-D cartesian} \Rightarrow \frac{\partial^2 V}{\partial x^2} = 0 \quad \dots(3)$$

$$\text{2-D cartesian} \Rightarrow \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0 \quad \dots(4)$$

$$\text{3-D cartesian} \Rightarrow \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0 \quad \dots(5)$$

$$\text{1-D polar} \Rightarrow \frac{\partial}{\partial r} \left(r^2 \frac{\partial V}{\partial r} \right) = 0 \quad \dots(6)$$

**Laplace equation in I-D cartesian:**

$$\frac{\partial^2 V}{\partial x^2} = 0$$

General solution $V(x) = mx + b$ (m and b are constant)

The solution is a straight line. The boundary condition of the specific problem fixes the constants (m and b).

Suppose it is given that $V=4$ at $x=1$ and $V=0$ at $x=5$

Using this two information you can find $m = -1$, $b = 5$.

Laplace equation in 2-D cartesian:

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$

We will solve the Laplace equation by separation of variables. You must have known the technique of separation of variable. Still we will revise briefly.

Laplace equation is, $\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$... (8)

Let's take $V(x, y)$ as a product of two functions each of them depends on x and y only

$$V(x, y) = X(x)Y(y) \quad \dots (9)$$

Putting $V(x, y)$ on Eq. (8) we get

$$Y \frac{\partial^2 X}{\partial x^2} + X \frac{\partial^2 Y}{\partial y^2} = 0$$

(as X & Y are function of x & y only we can replace partial derivative operator into normal derivative)

$$\Rightarrow Y \frac{d^2 X}{dx^2} + X \frac{d^2 Y}{dy^2} = 0$$

$$\Rightarrow \frac{1}{X} \frac{d^2 X}{dx^2} + \frac{1}{Y} \frac{d^2 Y}{dy^2} = 0$$

Now since each side depends on single variable only and X and Y are independent, we can take each side as constant.

$$\frac{1}{X} \frac{d^2 X}{dx^2} = -\frac{1}{Y} \frac{d^2 Y}{dy^2} = -k^2 \quad (k = \text{constant})$$

$$\Rightarrow \frac{d^2 X}{dx^2} = k^2 X \text{ and } \frac{d^2 Y}{dy^2} = -k^2 Y \quad \dots (10)$$

We will obtain the solution

$$\left. \begin{aligned} X &= Ae^{kx} + Be^{-kx} \\ Y &= C \sin ky + D \cos ky \end{aligned} \right\} \quad \dots (10)$$

Hence, $V(x, y) = (Ae^{kx} + Be^{-kx})(C \sin ky + D \cos ky)$

A, B, C, D are constants and depends on specific problems. You can of course take

$X = A \cos kx + B \sin kx$ and $Y = C e^{-ky} + D e^{ky}$



Laplace equation in 3D cartesian:

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0 \quad \dots (11)$$

Take $V(x, y, z) = X(x)Y(y)Z(z)$

Put in the equation and divide by xyz

$$\Rightarrow \frac{1}{X} \frac{d^2 X}{dx^2} + \frac{1}{Y} \frac{d^2 Y}{dy^2} + \frac{1}{Z} \frac{d^2 Z}{dz^2} = 0$$

We will take

$$\frac{d^2 X}{dx^2} = (k^2 + \ell^2)X, \quad \frac{d^2 Y}{dy^2} = -k^2 Y, \quad \frac{d^2 Z}{dz^2} = -\ell^2 Z$$

The solution will be

$$\left. \begin{aligned} X(x) &= A e^{\sqrt{k^2 + \ell^2} x} + B e^{-\sqrt{k^2 + \ell^2} x} \\ Y(y) &= C \sin ky + D \cos ky \\ Z(z) &= E \sin \ell z + F \cos \ell z \end{aligned} \right\} \quad \dots (12)$$

Laplace Equation in 3-D spherical coordinate :

Laplace equation in spherical coordinate can be written as

$$\frac{1}{r^2 \sin \theta} \left[\frac{\partial}{\partial r} \left(r^2 \sin \theta \frac{\partial}{\partial r} \right) + \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin \theta} \frac{\partial^2}{\partial \phi^2} \right] V = 0 \quad \dots (13)$$

Due to spherical symmetry potential V is function of r and θ independent of ϕ

$$\Rightarrow \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial V}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial V}{\partial \theta} \right) = 0 \quad \dots (14)$$

$$\Rightarrow \frac{\partial}{\partial r} \left(r^2 \frac{\partial V}{\partial r} \right) + \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial V}{\partial \theta} \right) = 0$$

Let $V(r, \theta) = R(r)S(\theta)$

Putting this in above equation and divided by $R(r)S(\theta)$, we get

$$\Rightarrow \frac{1}{R(r)} r^2 \frac{\partial^2 R(r)}{\partial r^2} + \frac{2r}{R(r)} \frac{\partial R(r)}{\partial r} + \frac{1}{S(\theta)} \frac{d}{d\theta} \left(\sin \theta \frac{dS}{d\theta} \right) = 0$$

$$\Rightarrow \frac{1}{R(r)} r^2 \frac{\partial^2 R(r)}{\partial r^2} + \frac{2r}{R(r)} \frac{\partial R(r)}{\partial r} = -\frac{1}{S(\theta)} \frac{d}{d\theta} \left(\sin \theta \frac{dS}{d\theta} \right) = n(n+1)$$

$$\therefore r^2 \frac{d^2 R(r)}{dr^2} + 2r \frac{dR(r)}{dr} - n(n+1)R(r) = 0 \quad \dots (15)$$

$$\text{And } \frac{d}{d\theta} \left(\sin \theta \frac{dS}{d\theta} \right) + n(n+1)S(\theta) = 0 \quad \dots (16)$$

Let $\cos \theta = x$



$$\Rightarrow (1-x^2) \frac{d^2 S}{dx^2} - 2x \frac{dS}{dx} + n(n+1)S = 0 \quad \dots (17)$$

It is Legendre's differential equation. Solution of the equation is Legendre's polynomial.

$$S(\theta) = P_\ell(\cos \theta)$$

$$\frac{d}{dr} \left(r^2 \frac{dR}{dr} \right) - n(n+1)R = 0 \quad \text{Let } r = e^u$$

$$\Rightarrow \frac{1}{e^u} \frac{d}{du} \left(e^u \frac{dR}{du} \right) - n(n+1)R = 0$$

$$\Rightarrow \frac{d^2 R}{du^2} + \frac{dR}{du} - n(n+1)R = 0 \quad \dots (18)$$

$$R = e^{mu}$$

$$\Rightarrow m^2 + m - n(n+1) = 0$$

$$\Rightarrow (m^2 - n^2) + (m - n) = 0$$

$$\Rightarrow (m - n)(m + n + 1) = 0$$

$$\Rightarrow m = n, m = -(n+1)$$

$$\therefore R(r) = Ae^{un} + Be^{-(n+1)u} = Ar^n + Br^{-(n+1)}$$

$$\therefore V(r, \theta) = R(r)S(\theta) = \sum_{n=0}^{\infty} \left(Ar^n + \frac{B}{r^{n+1}} \right) P_\ell(\cos \theta) \quad \dots (19)$$

Laplace equation in cylindrical coordinate

Due to symmetric about z-axis V is a function of r and θ . The Laplace equation in cylindrical coordinate is

$$\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial V}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 V}{\partial \theta^2} = 0 \quad \dots (20)$$

$$\text{Let } V(r, \theta) = F_1(r), F_2(\theta)$$

$$\Rightarrow \frac{1}{F_1(r)} \frac{1}{r} \frac{d}{dr} \left(r \frac{dF_1}{dr} \right) + \frac{1}{F_2} \frac{1}{r^2} \frac{d^2 F_2(\theta)}{d\theta^2} = 0$$

$$\Rightarrow \frac{r}{F_1(r)} \frac{d}{dr} \left(r \frac{dF_1}{dr} \right) + \frac{1}{F_2} \frac{d^2 F_2(\theta)}{d\theta^2} = 0$$

$$\Rightarrow \frac{r}{F_1(r)} \frac{d}{dr} \left(r \frac{dF_1}{dr} \right) = -\frac{1}{F_2} \frac{d^2 F_2(\theta)}{d\theta^2} = n^2$$

$$\therefore \frac{d}{dr} \left(r \frac{dF_1}{dr} \right) - \frac{n^2}{r} F_1 = 0 \quad \dots (21)$$



And
$$\frac{d^2 F_2(\theta)}{d\theta^2} + n^2 F_2(\theta) \quad \dots (22)$$

For $n=0$, we have solution,

$$F_1 = A_0 \ell n r + B_0 \quad \dots (23)$$

$$F_2 = C_0 \theta + D_0 \quad \dots (24)$$

For $n \neq 0$ putting $x = \ell n r$ equation (21), we get

$$\frac{d^2 F_1}{dx^2} - n^2 F_1 = 0$$

$$\therefore F_1 = A_n e^{nx} + B_n e^{-nx} = \left(A_n \cdot r^n + \frac{B_n}{r^n} \right) \quad \dots (25)$$

$$\text{Now, } F_2 = C_n \cos n\theta + D_n \sin n\theta \quad \dots (26)$$

Combining these solution we can write for the total potential

$$V_0 = (A_0 \ell n r + B_0)(C_0 \theta + D_0) \quad \text{when, } n=0$$

$$V_n = \left(A_n r^n + \frac{B_n}{r^n} \right) (C_n \cos n\theta + D_n \sin n\theta) \quad \text{when, } n \neq 0$$

Since V_n must have single valued i.e. we have

$$V_n(r, \theta) = V_n(r, 2\pi + \theta)$$

This shows that $C_0 = 0$

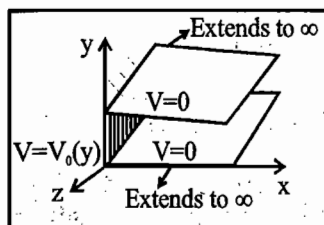
$$\therefore V_n = A_0 \ell n r + \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta) r^n + \sum_{n=1}^{\infty} (c_n \cos n\theta + d_n \sin n\theta) \frac{1}{r^n} + e_0 \quad \dots (27)$$

where, A_0, a_n, b_n, d_n, e_0 are constant.



SOLVED PROBLEMS

1. Two infinite ground metal plates lie parallel to the xz -plane, one at $y=0$, the other at $y=a$. The Left end, at $x=0$, is closed off with an infinite strip insulated from the two plates and maintained at a specific potential $V_0(y)$. Find the potential inside the 2 plates.



Soln. We know the solution of Laplace equation in 2-D cartesian as

$$V(x, y) = (Ae^{-ky} + Be^{ky})(C \sin ky + D \cos ky)$$

The problem has constraints in x -direction and symmetric in y (within the region). Then x will be exponential

The boundary conditions are

(i) $V = 0$ when $y = 0$

(ii) $V = 0$ when $y = a$

(iii) $V = V_0(y)$ when $x = 0$

(iv) $V \rightarrow 0$ as $x \rightarrow \infty$

Condition (iv) given $A = 0$ otherwise Ae^{-ky} will be infinity when $x \rightarrow \infty$

So, $V(x, y) = Be^{-ky}(C \sin ky + D \cos ky)$

Now a constant multiplied by another constant again a constant. So we write $BC = C$ and $BD = D$

$$V(x, y) = e^{-ky}(C \sin ky + D \cos ky)$$

Condition (i) now gives $D = 0$ otherwise V will be zero at $y = 0$

$$V(x, y) = Ce^{-ky} \sin ky$$

Now condition (ii) says $V = 0$ when $y = a$

$$\sin ka = 0 \Rightarrow ka = n\pi (n = 1, 2, 3 \dots \text{integer}) \Rightarrow k = \frac{n\pi}{a}$$

So we have the general solution

$$V(x, y) = \sum_{n=1}^{\infty} C_n e^{-\frac{n\pi x}{a}} \sin\left(\frac{n\pi y}{a}\right)$$

Then we know $V = V_0(y)$ at $x = 0$

So,
$$V(0, y) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi y}{a}\right) = V_0(y)$$

So by Fourier series we can get

$$C_n = \frac{2}{a} \int_0^a V_0(y) \sin\left(\frac{n\pi y}{a}\right) dy \quad (\text{Go back to Fourier series if you don't know})$$

The values of C_n 's will depend on $V_0(y) = V_0$



Then

$$C_n = \frac{2V_0}{a} \int_0^a \sin\left(\frac{n\pi y}{a}\right) dy$$

$$= \frac{2V_0}{n\pi} (1 - \cos n\pi)$$

0; if n is even

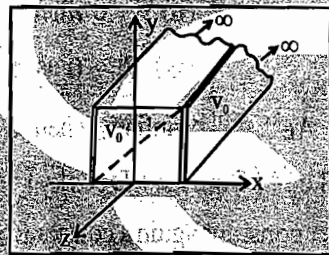
$$= \frac{4V_0}{n\pi}; \text{ if } n \text{ is odd}$$

So for a constant V_0 , the solution of given problem is

$$V(x, y) = \frac{4V_0}{\pi} \sum_{n=1,3,5} \frac{1}{n} e^{-\frac{n\pi x}{a}} \sin\left(\frac{n\pi y}{a}\right)$$

(As for even m the C_n 's are zero solution will non-zero for odd n's only).

2. Two infinitely long grounded metal plates again at $y=0$ and $y=a$, are connected at $x=\pm b$ by metal strips maintained at a constant potential V_0 (a thin layer of insulator prevents them from shorting out). Find the potential inside the rectangular pipe.



Soln. The boundary conditions are

(i) $V = 0$ when $y = 0$

(ii) $V = 0$ when $y = a$

(iii) $V = V_0$ when $x = b$

(iv) $V = V_0$ when $x = -b$

General Solution : $S \Rightarrow V(x, y) = (Ae^{+kx} + Be^{-kx})(C \sin ky + D \cos ky)$

This time we can't set $A = 0$. The region in it does not extend to $x = \infty$, so e^{kx} term will be there.

On the other hand the, the problem is symmetric with respect to x , so $V(-x, y) = V(x, y)$

This follows $Ae^{kx} + Be^{-kx} = V(x, y) = Ae^{-kx} + Be^{kx} = V(-x, y)$

(putting $x = -x$ in 1st term)

Hence $A = B$

$$\therefore V(x, y) = A(e^{kx} + e^{-kx})(C \sin ky + D \cos ky)$$

$$= 2A \cosh(kx)(C \sin ky + D \cos ky)$$

$$[e^{+kx} + e^{-kx} = 2 \cosh(kx)]$$

Again absorbing $2A$ into C and D

$$V(x, y) = \cosh(kx)(C \sin ky + D \cos ky)$$

from boundary condition (i) $\Rightarrow D = 0$

$$(ii) \Rightarrow \sin ka = 0$$



Hence $V(x, y) = C \cosh\left(\frac{n\pi x}{a}\right) \sin\left(\frac{n\pi y}{a}\right)$

as this satisfy for all positive integer value of n

$$\Rightarrow V(x, y) = \sum_{n=1}^{\infty} C_n \cosh\left(\frac{n\pi x}{a}\right) \sin\left(\frac{n\pi y}{a}\right)$$

And from boundary condition (iii)

$$\Rightarrow V(b, y) = \sum_{n=1}^{\infty} C_n \cosh\left(\frac{n\pi b}{a}\right) \sin\left(\frac{n\pi y}{a}\right) = V_0$$

$$C_n \cosh\left(\frac{n\pi b}{a}\right) = \frac{2}{a} \int_0^a V_0 \sin\left(\frac{n\pi y}{a}\right) dy$$

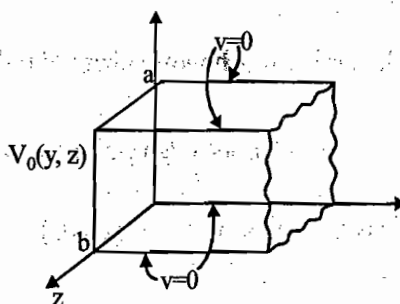
$$= \frac{2V_0}{a} \int_0^a \sin\left(\frac{n\pi y}{a}\right) dy \quad (\text{as } V_0 = \text{constant})$$

$$= \frac{2V_0}{n\pi} (1 - \cos n\pi) = \begin{cases} 0, & \text{if } n \text{ is even} \\ \frac{4V_0}{n\pi}, & \text{if } n \text{ is odd} \end{cases}$$

$$\Rightarrow C_n = \frac{4V_0}{n\pi} \frac{1}{\cosh\left(\frac{n\pi b}{a}\right)} \quad \text{for odd } n\text{'s}$$

$$\Rightarrow \text{The particular solution of this prob now, } V(x, y) = \frac{4V_0}{\pi} \sum_{n=1,3,5,\dots}^{\infty} \frac{\cosh\left(\frac{n\pi x}{a}\right)}{\cosh\left(\frac{n\pi b}{a}\right)} \sin\left(\frac{n\pi y}{a}\right)$$

8. An infinite long rectangular metal pipe of sides a and b is grounded. But one end at $x = 0$, there is a specific potential $V_0(y, z)$. Find the potential inside the pipe.



Soln. Boundary conditions are a

(i) $v = 0$ when $y = 0$

(iii) $v = 0$ when $z = 0$

(v) $V \rightarrow 0$ as $x \rightarrow \infty$

(ii) $v = 0$ when $y = b$

(iv) $v = 0$ when $z = a$

(vi) $V = V_0(y, z)$ when $x = 0$

The technique is the same as solving 2D problem



(v) gives $A=0$ (i) gives $D=0$, (iii) gives $F=0$

(ii) And (iv) requires $k = \frac{n\pi}{a}$ and $l = \frac{m\pi}{b}$, where n and m are position integers

Absorbing the constants we have

$$V(x, y, z) = Ce^{-\pi\sqrt{\left(\frac{n}{a}\right)^2 + \left(\frac{m}{b}\right)^2} x} \sin\left(\frac{n\pi y}{a}\right) \sin\left(\frac{m\pi z}{b}\right)$$

Most general soln (linear combination) is

$$V(x, y, z) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} C_{n,m} \exp\left\{-\pi\sqrt{\left(\frac{n}{a}\right)^2 + \left(\frac{m}{b}\right)^2} x\right\} \sin\left(\frac{n\pi y}{a}\right) \sin\left(\frac{m\pi z}{b}\right)$$

Condition (vi) gives

$$V(0, y, z) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} C_{n,m} \sin\left(\frac{n\pi y}{a}\right) \sin\left(\frac{m\pi z}{b}\right) = V_0(y, z)$$

Again from properties of Fourier series

$$C_{n,m} = \frac{4}{ab} \int_0^a \int_0^b V_0(y, z) \sin\left(\frac{n\pi y}{a}\right) \sin\left(\frac{m\pi z}{b}\right) dy dz$$

We take V_0 as constant then

$$C_{nm} = \frac{4V_0}{ab} \int_0^a \sin\left(\frac{n\pi y}{a}\right) dy \int_0^b \sin\left(\frac{m\pi z}{b}\right) dz = 0, \text{ if } n \text{ or } m \text{ is even}$$

$$\frac{16V_0}{\pi^2 nm}, \text{ if } n \text{ and } m \text{ are odd}$$

So the particular solution for constant V_0 is

$$V(x, y, z) = \frac{16V_0}{\pi^2} \sum_{n,m=1,3,5}^{\infty} \frac{1}{nm} \exp\left\{-\pi\sqrt{\left(\frac{n}{a}\right)^2 + \left(\frac{m}{b}\right)^2} x\right\} \sin\left(\frac{n\pi y}{a}\right) \sin\left(\frac{m\pi z}{b}\right)$$

4. The potential $V_0(\theta)$ is specified on the surface of a hollow sphere of radius R . Find the potential inside the sphere.

Soln. While finding potential inside the sphere, the $r=0$ is included within the region. So, B_l will be zero otherwise

$$\frac{B_l}{r^{l+1}} \text{ will diverge to infinity at } r=0. \text{ Thus } V(r, \theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos \theta)$$

At $r=R$; $V = V_0(\theta)$ specified by the prob.

$$V(R, \theta) = \sum_{l=0}^{\infty} A_l R^l P_l(\cos \theta) = V_0(\theta)$$

By properties of Legendre-polynomials (Refer to maths, physics)

$$\int_0^\pi P_l(\cos \theta) P_m(\cos \theta) \sin \theta d\theta = 0$$

$$0 = \text{if } \ell' \neq \ell = \frac{2}{2\ell+1} \quad \text{if } \ell' = \ell$$

Hence multiplied $V_0(\theta) = \sum A_\ell R^\ell P_\ell(\cos\theta)$ by $P_{\ell'}(\cos\theta) \sin\theta$ and integrating we have

$$\frac{2}{2\ell'+1} A_{\ell'} R^{\ell'} = \int_0^\pi V_0(\theta) P_{\ell'}(\cos\theta) \sin\theta d\theta$$

$$\Rightarrow A_\ell = \frac{2\ell+1}{2R^\ell} \int_0^\pi V_0(\theta) P_\ell(\cos\theta) \sin\theta d\theta \quad (\text{As } \ell = \ell' \text{ for non zero value})$$

Now $V_0(\theta)$ is not specified Let's take

$$V_0(\theta) = k \sin^2\left(\frac{\theta}{2}\right) \quad (\text{where } k \text{ is constant})$$

We rewrite this on

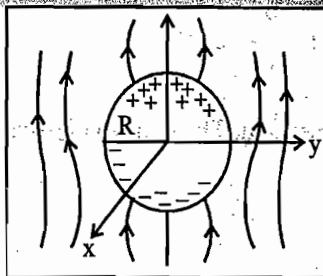
$$V_0(\theta) = \frac{k}{2} (1 - \cos\theta) = \frac{k}{2} [P_0(\cos\theta) - P_1(\cos\theta)]$$

So, $\sum_\ell A_\ell R^\ell P_\ell(\cos\theta) = \frac{k}{2} [P_0(\cos\theta) - P_1(\cos\theta)]$

just matching both side, $A_0 = \frac{k}{2}$, $A_1 = -\frac{k}{2R}$, Other A_ℓ 's vanish

$$\text{Hence the solution is, } V(r, \theta) = \frac{k}{2} \left[r^0 P_0(\cos\theta) - \frac{r}{R} P_1(\cos\theta) \right] = \frac{k}{2} \left(1 - \frac{r}{R} \cos\theta \right)$$

5. An uncharged metal sphere of radius R is placed in an otherwise uniform electric field $\vec{E} = E_0 \hat{z}$. Find the potential in the region outside the sphere



Soln. As from the basic properties of conductor, the surface of the sphere is equipotential (sub section 2.2.1) we set the potential at surface is zero

The field will push positive charge to the northern surface of the sphere leaving a negative charge in the southern sphere. The induced charge will distort the field in the neighbourhood of the sphere, but at long distance the field will be unchanged.

So far from the sphere, $V \rightarrow -E_0 z + C$

Since $V = 0$ at $z = 0$ then $c = 0$

so the boundary condition of the problem are

- (i) $v = 0$ at $r = R$ (ii) $v = -E_0 z = -E_0 r \cos\theta$ for $r \gg R$



the general solution is $V(r, \theta) = \sum_{\ell} \left(A_{\ell} r + \frac{B_{\ell}}{r^{\ell+1}} \right) P_{\ell}(\cos \theta)$

Condition (i) given (put $r = R$, and $v = 0$ at $r = R$)

$$A_{\ell} R + \frac{B_{\ell}}{R^{\ell+1}} = 0$$

Or

$$B_{\ell} = -A_{\ell} R^{2\ell+1}$$

So,

$$V(r, \theta) = \sum_{\ell=0}^{\infty} A_{\ell} \left(r^{\ell} - \frac{R^{2\ell+1}}{r^{\ell+1}} \right) P_{\ell}(\cos \theta)$$

For $r \gg R$ $V(r, \theta)$ must be $-E_0 r \cos \theta$

This requires, $\left(\frac{R^{2\ell+1}}{r^{\ell+1}} \rightarrow 0 \text{ at } r \rightarrow \infty \right)$

So,

$$\sum A_{\ell} r^{\ell} P_{\ell}(\cos \theta) = -E_0 r \cos \theta$$

At right side only one term is present ($P_1(\cos \theta) = \cos \theta$)

$$\sum A_{\ell} r^{\ell} P_{\ell}(\cos \theta) = -E_0 r P_1(\cos \theta)$$

$$\text{Then, } \left(A_0 + A_1 r \cos \theta + A_2 r^2 \left(\frac{3 \cos^2 \theta - 1}{2} \right) + \dots \right) = -E_0 r \cos \theta$$

Hence only A_1 is present, out other A_{ℓ} 's are zero.

$$A_1 = -E_0$$

So

$$V(r, \theta) = -E_0 \left(r - \frac{R^3}{r^2} \right) \cos \theta$$

The first term $-E_0 r \cos \theta$ is due to external field. The contribution from the induced charge is

$$\frac{E_0 R^3}{r^2} \cos \theta$$

This is indeed the potential of a dipole of dipole moment $\boxed{p = E_0 R^3}$.



PRACTICE SET

1. One of electrostatic boundary conditions for the potential v for a surface carrying a surface charge density s can be written as

$$(a) V_{\text{above}} - V_{\text{below}} = -\frac{\sigma}{\epsilon_0}$$

$$(b) V_{\text{above}} - V_{\text{below}} = \frac{\sigma}{\epsilon_0}$$

$$(c) \frac{\partial V_{\text{above}}}{\partial n} - \frac{\partial V_{\text{below}}}{\partial n} = \frac{\sigma}{\epsilon_0}$$

$$(d) \frac{\partial V_{\text{above}}}{\partial n} - \frac{\partial V_{\text{below}}}{\partial n} = -\frac{\sigma}{\epsilon_0}$$

2. In an electrostatic boundary value problem, the surface charge density σ is specified at $x = 0$. The corresponding boundary condition for potential V can be written as

$$(a) \left(\frac{\partial^2 V}{\partial x^2} \right)_{x=0} = \frac{1}{\epsilon_0} \sigma(y, z)$$

$$(b) \left(\frac{\partial^2 V}{\partial x^2} \right)_{x=0} = -\frac{1}{\epsilon_0} \sigma(y, z)$$

$$(c) \left(\frac{\partial V}{\partial x} \right)_{x=0} = +\frac{1}{\epsilon_0} \sigma(y, z)$$

$$(d) \left(\frac{\partial V}{\partial x} \right)_{x=0} = -\frac{1}{\epsilon_0} \sigma(y, z)$$

3. The electrostatic potential is given by, $\Phi = 0$ at $x = 0$ and $x = a$, $\Phi = V$ $0 \leq x \leq a$.

Then Φ is given by $\Phi = \sum_{n=1}^{\infty} A_n \sin(\alpha x)$ where α is:

$$(a) \frac{n\pi}{2}$$

$$(b) \frac{n\pi}{a}$$

$$(c) \frac{na}{\pi}$$

$$(d) \frac{na}{2\pi}$$

4. Solving Poisson's equation $\nabla^2 \phi = -\rho_0 / \epsilon_0$ for the electrostatic potential $\phi(\vec{x})$ in a region with a constant charge density ρ_0 , two students find different answers, viz,

$$\phi_1(x) = -\frac{1}{2} \frac{\rho_0 x^2}{\epsilon_0} \text{ and } \phi_2(x) = -\frac{1}{2} \frac{\rho_0 y^2}{\epsilon_0}$$

The reason why these different solutions are both correct is because

(a) space is isotropic and hence x and y are physically equivalent.

(b) We can add solutions of Laplace's equation to both $\phi_1(\vec{x})$ and $\phi_2(\vec{x})$

(c) The electrostatic energy is infinite for a constant charge density.

(d) The boundary conditions are different in the two cases.

5. A sphere made of a material of infinite permittivity and placed in a uniform electric field behaves like a

(a) conducting sphere,

(b) free space

(c) perfect dielectric sphere

(d) none of these

6. Electric fields $\vec{E}_1$ and $\vec{E}_2$ on the two sides of an interface of two dielectric media having dielectric constants K_1 and K_2 , make angles θ_1 and θ_2 with the normal to the interface. Which one of the following relations is true?

$$(a) K_1 \sin \theta_1 = K_2 \sin \theta_2$$

$$(b) K_1 \tan \theta_1 = K_2 \tan \theta_2$$



7. The function $P_n(x)$ is even if n is
 (a) even (b) zero only (c) odd (d) none of these
8. A function is said to be harmonic if it satisfies:
 (a) Hermite differential equation (b) Laplace's equation
 (c) Legendre's differential equation (d) Wave equation
9. Circular harmonics are the solution of Laplace's equation in
 (a) plane polar coordinates (b) cartesian coordinates
 (c) spherical polar coordinates (d) none of these
10. Two potential functions ϕ_1 and ϕ_2 satisfy Laplace's equation within a closed volume and assume the same values on its surface. Then
 (a) $\phi_1 = \phi_2$ (b) $\phi_1 - \phi_2 \neq 0$
 (c) ϕ_1 may or may not be equal to ϕ_2 (d) none of these

ANSWER KEY					
Questions	1	2	3	4	5
Option	(d)	(d)	(b)	(d)	(a)
Questions	6	7	8	9	10
Option	(d)	(a)	(b)	(a)	(a)

CAREER ENDEAVOUR

Chapter 7

Electric Image Method

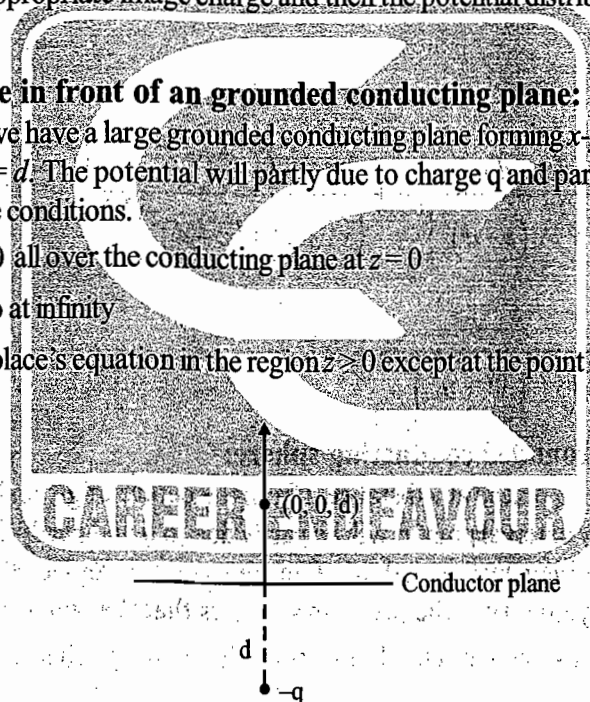
Introduction:

There is a class of electrostatic problems with boundary condition that appear to be difficult to satisfy if the governing Laplace's equation is to be solved directly but the condition on the bounding surfaces in these problems can be set up by appropriate image charge and then the potential distribution can be determined in straight forward manner.

(1) Point charge in front of an grounded conducting plane:

Suppose we have a large grounded conducting plane forming $x-y$ plane and a positive charge placed on the z -axis at $z = d$. The potential will partly due to charge q and partly due to induce charge $-q$. In this process it obey the conditions.

- (1) Potential $\phi = 0$ all over the conducting plane at $z = 0$
- (2) ϕ must be zero at infinity
- (3) ϕ satisfies Laplace's equation in the region $z > 0$ except at the point where the charge q is located.



From the above condition we can write the image charge $-q$ located at $z = -d$.

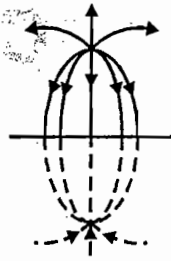
Potential and field at any point $P(x, y, z)$:

$$\phi(x, y, z) = \frac{1}{4\pi\epsilon_0} \left(\frac{q}{\sqrt{x^2 + y^2 + (z-d)^2}} - \frac{q}{\sqrt{x^2 + y^2 + (z+d)^2}} \right) \quad \dots (1)$$

and the field

$$\vec{E} = -\nabla\phi = -\left[\frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k} \right] \quad \dots (2)$$

Line of force:



Force on +q:

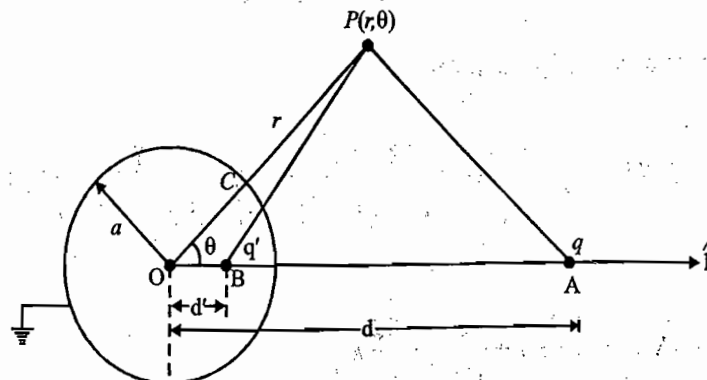
$$\vec{F} = \frac{qq'}{4\pi\epsilon_0(2d)^2}(-\hat{k}) = -\frac{q^2}{16\pi\epsilon_0 d^2}\hat{k} \quad \dots (3)$$

Induce surface charge density:

$$\begin{aligned} \sigma &= -\epsilon_0 \left. \frac{\partial \phi}{\partial z} \right|_{z=0} \\ &= \frac{q}{4\pi} \left[\frac{(z-d)}{\{x^2 + y^2 + (z-d)^2\}^{3/2}} - \frac{(z+d)}{\{x^2 + y^2 + (z+d)^2\}^{3/2}} \right]_{z=0} \\ \therefore \sigma(x, y) &= -\frac{q}{2\pi} \frac{d}{(x^2 + y^2 + d^2)^{3/2}} \quad \dots (4) \end{aligned}$$

(2) A Point charge and a grounded conducting sphere:

Consider a point charge q placed at a distance d from the centre of grounded conducting sphere of radius a . The potential at any external point P will be partly due to q and partly due to induced charges on the sphere. To solve the problem by the method of image we are to replace the induced charges by suitable image charges. The symmetry of the problem suggests that the image charge must be placed symmetrically about the line of symmetry OA . Let us try a single image charge q' located within the sphere at some point B (with $OB = d'$) on the line OA . In this attempt the following conditions must be obeyed.



(i) Potential ϕ is zero all over the surface of the sphere.

(ii) Potential ϕ must be zero at infinity

Let all possible points excepting at A

Now the potential at any point $C(a, \theta)$ on the surface of the sphere must be zero i.e.

$$\frac{1}{4\pi\epsilon_0} \left[\frac{q}{AC} + \frac{q'}{BC} \right] = 0 \quad \dots (5)$$

$$\text{Or, } \frac{q}{\sqrt{a^2 + d^2 - 2ad \cos \theta}} + \frac{q'}{\sqrt{a^2 + d'^2 - 2ad' \cos \theta}} = 0$$

$$\text{Or, } \frac{q/a}{\sqrt{1 + \frac{d^2}{a^2} - 2\frac{d}{a} \cos \theta}} = \frac{-q'/d'}{\sqrt{1 + \frac{a^2}{d'^2} - 2\frac{a}{d'} \cos \theta}}$$

To make this equality valid for all values of θ we may choose

$$\frac{q}{a} = -\frac{q'}{d'} \text{ and } \frac{d}{a} = \frac{a}{d'}$$

$$\text{Therefore, } q' = -\frac{qa}{d} \text{ and } d' = \frac{a^2}{d} \quad \dots (6)$$

Potential and field at any point $P(r, \theta)$:

$$\begin{aligned} \phi(r, \theta) &= \frac{q}{4\pi\epsilon_0 AP} + \frac{1}{4\pi\epsilon_0} \frac{q'}{BP} \\ \Rightarrow \phi(r, \theta) &= \frac{1}{4\pi\epsilon_0} \left[\frac{q}{\sqrt{r^2 + d^2 - 2rd \cos \theta}} + \frac{q'}{\sqrt{r^2 + d'^2 - 2rd' \cos \theta}} \right] \quad \dots (7) \end{aligned}$$

where q' and d' are given by

Corresponding field is given by

$$\vec{E} = -\nabla\phi = -\left[r \frac{\partial\phi}{\partial r} + \theta \frac{1}{r} \frac{\partial\phi}{\partial\theta} \right] \quad \dots (8)$$

Therefore, the components of $\vec{E}$ are:

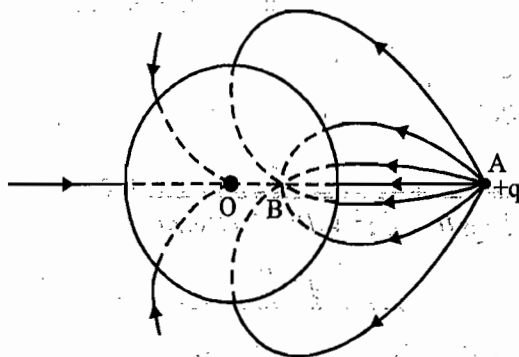
$$E_r = -\frac{\partial\phi}{\partial r} = \frac{1}{4\pi\epsilon_0} \left[\frac{q(r - d \cos \theta)}{(r^2 + d^2 - 2rd \cos \theta)^{3/2}} + \frac{q'(r - d' \cos \theta)}{(r^2 + d'^2 - 2rd' \cos \theta)^{3/2}} \right] \quad \dots (9)$$

$$E_\theta = -\frac{1}{r} \frac{\partial\phi}{\partial\theta} = \frac{1}{4\pi\epsilon_0} \left[\frac{qd \sin \theta}{(r^2 + d^2 - 2rd \cos \theta)^{3/2}} + \frac{q'd' \sin \theta}{(r^2 + d'^2 - 2rd' \cos \theta)^{3/2}} \right] \quad \dots (10)$$

Lines of force:

Lines of force will be the same as that between q at A and $q' = -qa/d$ at B . The lines meet the conducting sphere normally. The lines of force within the sphere are shown by dotted lines because field inside the sphere is zero and they do not exist really.

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**Force on +q:**

Force between +q at A and the sphere will be given by the force between +q at A and the image charge q' placed at B. Therefore, force on +q due to the sphere would be

$$\begin{aligned}\vec{F} &= \frac{1}{4\pi\epsilon_0} \cdot \frac{qq'}{(BA)^2} \hat{k} \quad \text{where } \hat{k} \text{ is a unit vector along OA} \\ &= -\frac{1}{4\pi\epsilon_0} \cdot \frac{q^2 a/d}{(d - a^2/d)^2} \hat{k} \quad \left[\because BA = OA - OB = d - a^2/d \right] \\ \Rightarrow \quad \vec{F} &= -\frac{q^2 ad}{4\pi\epsilon_0 (d^2 - a^2)^2} \hat{k} \quad \dots (11)\end{aligned}$$

The force is attractive in nature.

If the charge q is very close to the sphere, i.e., $d = a + \delta$ ($\delta \ll a$), then

$$\vec{F} \rightarrow -\frac{1}{4\pi\epsilon_0} \frac{q^2}{(2\delta)^2} \hat{k} \quad \dots (12)$$

This result is similar to the case of point charge placed in front of a conducting plane.

Induced charges:

The surface density of induced charges at any point $C(a, \theta)$ on the sphere is given by

$$\sigma(\theta) = -\epsilon_0 \left. \frac{\partial \phi}{\partial r} \right|_{r=a} = -\frac{q}{4\pi a} \frac{d^2 - a^2}{(a^2 + d^2 - 2ad \cos \theta)^{3/2}} \quad \dots (13)$$

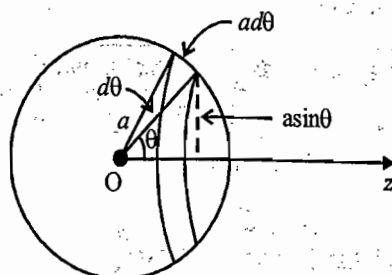
As expected the induced charge density is negative. Note that $(a^2 + d^2 - 2ad \cos \theta)^{3/2} = AC^3$. Thus, we can say that charge density at a point is inversely proportional to the cube of the distance of the point from q . The charge density is thus maximum at $\theta = 0^\circ$ and minimum at $\theta = 180^\circ$.

$$\sigma_{\max} = -\frac{q}{4\pi a} \frac{d+a}{(d-a)^2} \quad \text{and} \quad \sigma_{\min} = -\frac{q}{4\pi a} \frac{d-a}{(d+a)^2} \quad \dots (14)$$

Note that σ cannot be zero at any point on the sphere.

Total induced charge on the sphere can be obtained by integrating the charge density over the

$$Q = \int_0^\pi \sigma(\theta) \cdot 2\pi a \sin \theta \cdot a d\theta = -\frac{q(d^2 - a^2)a}{2} \int_0^\pi \frac{\sin \theta d\theta}{(a^2 + d^2 - 2ad \cos \theta)^{3/2}}$$



Putting $a^2 + d^2 - 2ad \cos \theta = p^2$ we get

$$Q = -\frac{q(d^2 - a^2)}{2d} \int_{d-a}^{d+a} \frac{p dp}{p^3} = -\frac{q(d^2 - a^2)}{2d} \left[\frac{1}{p^2} \right]_{d-a}^{d+a} = \frac{qa}{d}$$

which is, as expected, equal to the image charge.

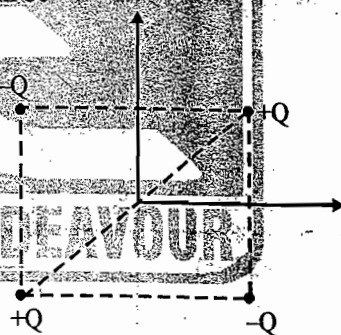
Number of Image:

Suppose a point charge q located at front of two semi infinite conductor plane and the angle between two plane is θ then the number of image charge will be

$$n = \left(\frac{360}{\theta} - 1 \right)$$

Example : for $\theta = 90^\circ$ and $n = 3$

$$n = \left(\frac{360}{90} - 1 \right) = (4 - 1) = 3$$



SOLVED PROBLEMS

A point charge q of mass m is released rest at a distance d from an infinite grounded conducting plate. Find the time taken by the charge to hit plate

ln. When the charge q is at a distance x from the plate the force on it due to the conducting plate is given by the force between q and its image charge $-q$ located at $-x$.

So, the equation of motion is given by

$$m \cdot \frac{d^2 x}{dt^2} = -\frac{1}{4\pi \epsilon_0} \cdot \frac{q^2}{(2x)^2}$$

Multiplying both sides by $2 \frac{dx}{dt}$ and integrating we get

$$m \left(\frac{dx}{dt} \right)^2 = -\frac{q^2}{8\pi \epsilon_0} \int \frac{dx}{x^2} = \frac{q^2}{8\pi \epsilon_0} \cdot \frac{1}{x} + c$$

where, c is a constant.



$$m \left(\frac{dx}{dt} \right)^2 = \frac{q^2}{8\pi \epsilon_0} \left(\frac{1}{x} - \frac{1}{d} \right) \text{ or } dt = - \sqrt{\frac{8\pi \epsilon_0 m}{q^2}} \frac{dx}{\sqrt{\frac{1}{x} - \frac{1}{d}}}$$

Integrating we get the required time as

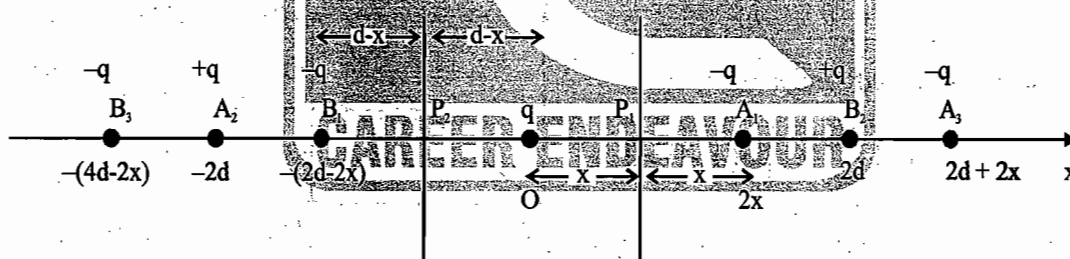
$$T = \int dt = - \sqrt{\frac{8\pi \epsilon_0 m}{q^2}} \int_d^0 \frac{dx}{\sqrt{\frac{1}{x} - \frac{1}{d}}} = \sqrt{\frac{8\pi \epsilon_0 m d^3}{q^2}} \int_0^d \frac{\sqrt{x}}{\sqrt{d-x}} dx$$

Substituting $x = d \sin^2 \theta$ we find that $\int_0^d \frac{\sqrt{x}}{\sqrt{d-x}} dx = d \frac{\pi}{2}$

$$\therefore T = \sqrt{\frac{8\pi \epsilon_0 m d^3}{q^2}} \cdot \frac{d\pi}{2} = \frac{\pi}{q} \sqrt{2\pi \epsilon_0 m d^3}$$

2. Two infinite parallel grounded conducting plates are held a distance d apart and a point charge q is placed between them. Find the location of image charges. Express the force on q by an infinite series. What is the magnitude of the force for $x = \frac{d}{2}$? Discuss the case $d \rightarrow \infty$.

Soln. Positions of the image are shown figure below. The charge q is assumed to be at the origin O . The charge $-q$ at A_1 is image of q due to plane 1 and $-q$ at B_1 is that due to plate 2. So, $OA_1 = 2x$ and $OB_1 = -2(d-x)$



Now as A_1 is in front of plate 2 we are to consider its image $+q$ at A_2 , where $A_2P_2 = A_1P_2 = 2x + d - x = d + x$. Therefore, $OA_2 = A_2P_2 + OP_2 = d + x + d - x = 2d$. Similarly, as B_1 is in front of plate 1 we are to consider its image at B_2 such that $B_1P_1 = P_1B_2 = 2(d-x) + x$. Therefore, $OB_2 = x + 2(d-x) + x = 2d$. Proceeding in this way we can find other image charges. Now force on q at O is given by the resultant force due to all the image charges. Thus,

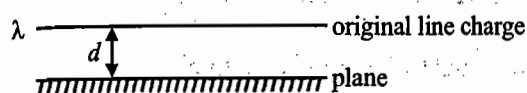
$$\vec{F} = \hat{x} \frac{q^2}{4\pi \epsilon_0} \left[\frac{1}{(2x)^2} - \frac{1}{(2d)^2} + \frac{1}{(2d+2x)^2} - \dots - \frac{1}{(2d-2x)^2} + \frac{1}{(2d)^2} - \frac{1}{(4d-2x)^2} + \dots \right]$$

If $x = \frac{d}{2}$ then $\vec{F} = 0$

If $d \rightarrow \infty$ then $\vec{F} = \hat{x} \frac{1}{4\pi \epsilon_0} \cdot \frac{q^2}{(2x)^2}$, same as for only one plane.



3. Consider uniform line charged λ is placed on an infinite straight wire, a distance d above a grounded conducting plane. The conducting plane is the xy plane. Find the potential in the region above the plane.



Soln.

$-\lambda$ ----- image line charge

The potential at P due to the original wire is

$$\phi_1(yz) = \frac{\lambda}{2\pi\epsilon_0} \ln \left(\frac{r_0}{r_1} \right)$$

due to image wire

$$\phi_2(yz) = -\frac{\lambda}{2\pi\epsilon_0} \ln \left(\frac{r_0}{r_2} \right)$$

Here r_0 is the reference point, where potential is zero.

Therefore, the total potential at P is

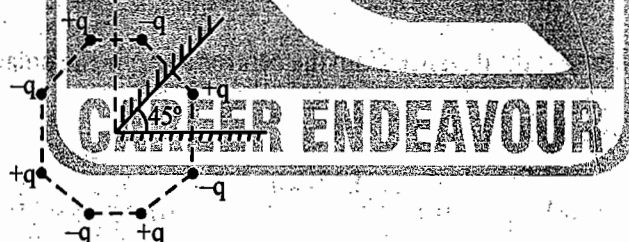
$$\phi(y, z) = \frac{\lambda}{2\pi\epsilon_0} \ln \left(\frac{r_2}{r_1} \right) = \frac{\lambda}{2\pi\epsilon_0} \ln \left\{ \frac{y^2 + (z+d)^2}{y^2 + (z-d)^2} \right\}$$

placed
is the

4. If the angle between to conducting plane 45° then find number of images and draw their position.

harge
 $\partial B_1 =$

Soln. $n = \left(\frac{360^\circ}{45^\circ} - 1 \right) = (8 - 1) = 7$



$d-x =$
are to
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PRACTICE SET

- A point charge is at a distance d from a metal plate. The force acting on the charge is:
 - Proportional to $1/d^2$ and repulsive.
 - zero
 - Proportional to $1/d^2$ and attractive.
 - Proportional to $1/d$ and attractive.
- Let a point charge $+q$ be at a distance d from an infinite conducting plate which is grounded.

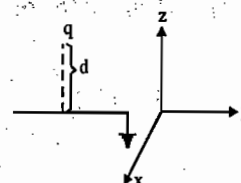
The force experienced by the charge is given by

(a) $\vec{F} = +\frac{1}{4\pi\epsilon_0} \frac{q^2}{d^2} \hat{k}$

(b) $\vec{F} = -\frac{1}{4\pi\epsilon_0} \frac{q^2}{d^2} \hat{k}$

(c) $\vec{F} = -\frac{1}{4\pi\epsilon_0} \frac{q^2}{(2d)^2} \hat{k}$

(d) $\vec{F} = +\frac{1}{4\pi\epsilon_0} \frac{q^2}{(2d)^2} \hat{k}$

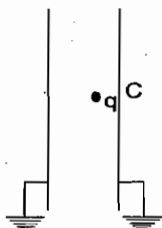


- Two equal and opposite charges are placed at a distance r from each other. A large metallic sheet is placed at a distance d from them as shown in figure. Due to presence of the sheet, the attractive force between the charges along the direction joining them
 - decreases
 - Increases
 - Remains unchanged.
 - Decreases if $r > d/2$ and increases if $r < d/2$.
- A conducting sphere of radius R is placed in a uniform electric field $\vec{E}_0$ directed along $+z$ axis. The electric potential for outside points is given as

$V_{\text{out}} = -E_0 \left(1 - \frac{R^3}{r^3} \right) r \cos \theta$, where r is the distance from the centre and θ is the polar angle. The charge density on the surface of the sphere is

- $3\epsilon_0 E_0 \cos \theta$
- $\epsilon_0 E_0 \cos \theta$
- $\frac{3\epsilon_0 E_0 \cos \theta}{2}$
- $\frac{\epsilon_0}{3} E_0 \cos \theta$

- A point charge q is kept at the mid-point between two large parallel grounded conducting plates. Assume no gravity. The charge is displaced a little towards the right plate. The charge will now,



- Stay where it is
- Move towards the right plate
- Move towards the left plate
- Oscillate between the plates

ANSWER KEY

Questions	1	2	3	4	5
Option	(c)	(c)	(a)	(a)	(b)

Chapter 8

Motion of charged particles

Introduction:

The study of the motion of charged particles in specified field is important since it provides a good physical insight for understanding some of the dynamical processes in plasmas. It also permits to obtain information on some macroscopic phenomena which are due to the collective behaviour of large number of particles. Not all of the components of the detailed microscopic particle motion contribute to macroscopic effect but it is possible to isolate the components of the individual motion that contribute to the collective plasma behaviour.

Lorentz force:

We know that when a charged particle moving in presence of electric field it feels a force $q\vec{E}$. Similarly, in the presence of a magnetic field $\vec{B}$, a charge q experiences an additional force.

$$\vec{F}_m = q\vec{v} \times \vec{B} \quad (\text{where } \vec{v} \text{ is the velocity}) \quad \dots (1.a)$$

Note

- (1) There is no force on a charge at rest.
- (2) A force is exerted on the charge only if there is component of the magnetic field perpendicular to the direction of the velocity.
- (3) $\vec{v} \cdot \vec{F}_m = 0$, which shows that the magnetic force does not do any work.

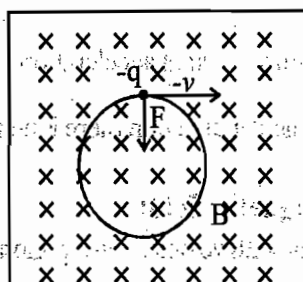
In case where both the electric and magnetic field are present, the force on the charge particle is given by

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B}) \quad \dots (1.b)$$

Is called Lorentz force

Cyclotron principle

A negatively charged particle introduced with velocity $\vec{v}$ into a uniform magnetic field $\vec{B}$ pointing downwards. Taking charge of the charged particle as $-q$ and the $\vec{B}$ and $\vec{v}$ as in right angles, describe the motion in below figure.





$$\vec{F} = Q(0 + \vec{v} \times \vec{B}) = -q(\vec{v} \times \vec{B})$$

Now, $\vec{v}$ is pointing right hand side and magnetic field $\vec{B}$ pointing downwards, $\vec{v} \times \vec{B}$ is upwards.

So, $-(\vec{v} \times \vec{B})$ will point downwards.

So, throughout the motion the $\vec{F}$ force will point perpendicular to the velocity of charged particle. So, the motion will be circular.

$$\vec{F} = -q(\vec{v} \times \vec{B})$$

$$F = qvB$$

This will equal to centrifugal force for steady motion.

$$F = qvB = \frac{mv^2}{r}$$

Or,

$$r = \frac{mv}{qB} \quad (1.c)$$

Angular velocity, $\omega = \frac{v}{r} = \frac{qB}{m} \quad (1.d)$

Frequency, $f = \frac{\omega}{2\pi} = \frac{qB}{2\pi m} \quad (1.e)$

So, a charged particle is introduced to a magnetic field with same initial velocity perpendicular to the field $\vec{B}$, the particle will perform circular motion. **This is Cyclotron Principle.**

Motion of charge particle in uniform electric field :

Suppose a charge particle of mass m and charge q moving in presence of uniform electric field $\vec{E}$, then we can write

$$m \frac{d^2 r}{dt^2} = qE \quad \dots (1.f)$$

$$\Rightarrow \frac{dr}{dt} = \frac{qEt}{m} + C$$

Let the particle has initial velocity v_0 and position r_0 .

$$\Rightarrow \frac{dr}{dt} = \frac{qEt}{m} + v_0$$

$$\Rightarrow r(t) = \frac{1}{2} \left(\frac{qE}{m} \right) t^2 + v_0 t + r_0$$

Therefore, the charge particle moving with a constant acceleration $\left(\frac{qE}{m} \right)$, in the direction of E if $a > 0$ and in the opposite direction if $a < 0$. In perpendicular direction there is no motion.

Motion of particle in uniform magnetic field:

For a particle of charge q mass m , moving with velocity $\vec{v}$ in a region of space where there is only a magnetic



$$m \frac{d\vec{v}}{dt} = q(\vec{v} \times \vec{B}) \quad \dots (3)$$

Now v can be separated in components parallel ($v_{\parallel}$) and perpendicular ($v_{\perp}$) to the magnetic field.

$$\vec{v} = v_{\parallel} + v_{\perp} \quad \dots (4)$$

putting it in equation (3), we get

$$m \frac{dv_{\parallel}}{dt} + m \frac{dv_{\perp}}{dt} = q(v_{\perp} + B) + q(\vec{v}_{\parallel} \times \vec{B})$$

The parallel component equation can be written as

$$\frac{dv_{\parallel}}{dt} = 0 \quad \dots (5)$$

$$\Rightarrow v_{\parallel} = v_0 \text{ (the initial velocity)}$$

It shows that the particle velocity along B does not change.

The perpendicular component equation is

$$\frac{dv_{\perp}}{dt} = \frac{a}{m} (\vec{v}_{\perp} \times \vec{B}) \quad \dots (6)$$

Let B is along z -direction

$$\therefore \vec{B} = B \hat{z}$$

$$v_{\parallel} = v_z$$

$$\vec{v}_{\perp} = v_x \hat{i} + v_y \hat{j}$$

Therefore, from equation (5) and (6), we get

$$\frac{dv_z}{dt} = 0 \quad \dots (7)$$

$$\Rightarrow v_z = v_0 \text{ (constant)} \quad \dots (7)$$

$$\text{And } \frac{d}{dt} (v_x \hat{i} + v_y \hat{j}) = \frac{a}{m} [(v_x \hat{i} + v_y \hat{j}) \times B \hat{z}]$$

$$\Rightarrow \frac{dv_x}{dt} \hat{i} + \frac{dv_y}{dt} \hat{j} = \frac{a}{m} B_0 (v_y \hat{i} - v_x \hat{j})$$

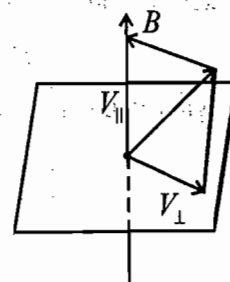
$$\Rightarrow \frac{dv_x}{dt} = \left(\frac{qB_0}{m} \right) v_y = \Omega_c v_y \quad \dots (8)$$

$$\frac{dv_y}{dt} = - \left(\frac{qB_0}{m} \right) v_x = -\Omega_c v_x \quad \dots (9)$$

From equation (8), we can write

$$\frac{d^2 v_x}{dt^2} + \Omega_c^2 v_x = 0 \quad \dots (10)$$

The solution of equation (10) is



So,

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we can

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magnetic



depends on the relation between the initial velocities. $v_x(0)$ and $v_y(0)$ according to

$$\tan(\theta_0) = \frac{v_y(0)}{v_x(0)}$$

Putting the equation (11) in equation (8), we get

$$v_y(t) = v_{\perp} \cos(\Omega_c t + \theta_0) \quad \dots (12)$$

Integrating the equation (11) and (12), we get

$$x(t) = -\frac{v_{\perp}}{\Omega_c} \cos(\Omega_c t + \theta_0) + x_0 \quad \dots (13)$$

$$y(t) = \frac{v_{\perp}}{\Omega_c} \sin(\Omega_c t + \theta_0) + y_0 \quad \dots (14)$$

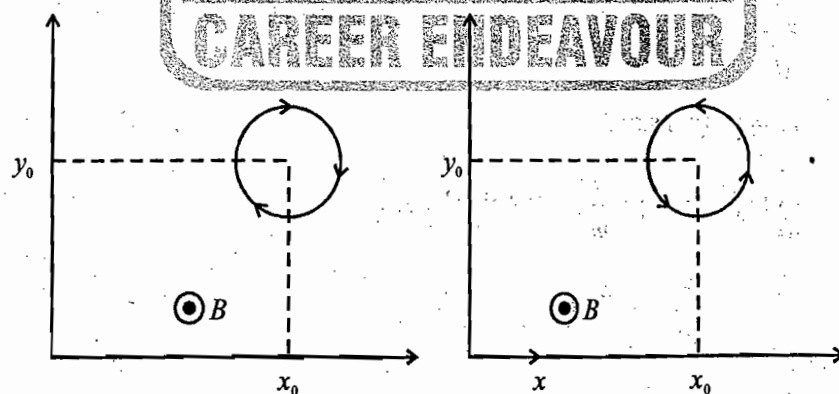
where, x_0, y_0 are the initial position.

From equation (13) and (14), we get

$$(x - x_0)^2 + (y - y_0)^2 = \left(\frac{v_{\perp}}{\Omega_c}\right)^2 = r_c^2 \quad \dots (15)$$

This shows that particle is rotating in circular path of radius $r_c = \left(\frac{v_{\perp}}{\Omega_c}\right)$

$$r_c = \frac{v_{\perp} m}{qB} \quad \dots (16)$$



The time period of a revolution in the circular orbit is

$$T = \frac{2\pi r}{v_{\perp}} = \frac{2\pi m v_{\perp}}{qB v_{\perp}}$$

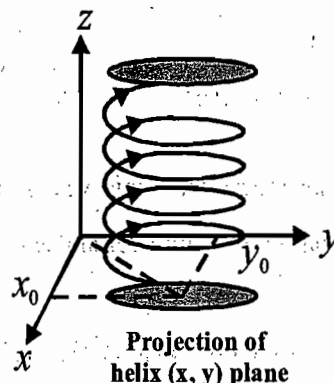
$\Rightarrow$

$$T = \frac{2\pi m}{qB} \quad \dots (17)$$

If there have no initial velocity component along v_z then particle will be in x-y plane but if $v_z \neq 0$ then



This represent a helix motion,



3. Magnetic Moment:

Suppose the particle has no velocity along z-direction, then it will be rotating in the circular path. To circular motion of a charged particle in a magnetic field there is associated a circulating electric current I . Since current is flowing in clockwise direction. Direction for a B field pointing towards the observer. From Ampere's law we can say it will produce a magnetic field opposite to the external field.

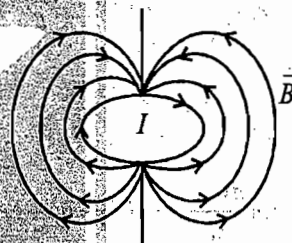
The magnetic moment M associated with the circulating current is normal to the area A bounded by the particle orbit.

$$\therefore |M| = (\text{current}) \cdot (\text{orbital area}) = IA$$

The current I is given by $I = \frac{q}{T_c} = \frac{q \cdot qB}{2\pi m} = \frac{q^2 B}{2\pi m}$

$$\therefore |M| = \frac{q^2 B}{2\pi m} \cdot \pi r_c^2 = \frac{1}{2} \frac{mv_1^2}{B}$$

$$\Rightarrow |M| = \frac{1}{2} \frac{mv_1^2}{B} \quad \dots (19)$$



4. Uniform electrostatic and Magnetic fields:

Formal solution:

We consider now the motion of a charged particle in the presence of both electric and magnetic fields which are constant in time and spaitiably uniform. The equation of motion is given by

$$m \frac{dV}{dt} = q(\vec{E} + \vec{v} \times \vec{B}) \quad \dots (20)$$

Taking components parallel and perpendicular to B .

$$V = v_{\parallel} + v_{\perp} \quad \dots (21)$$

$$E = E_{\parallel} + E_{\perp} \quad \dots (22)$$

We can resolve into two component equations

$$m \frac{dv_{\parallel}}{dt} = qE_{\parallel} \quad \dots (23)$$

$$m \frac{dv_{\perp}}{dt} = q(E_{\perp} + \vec{v}_{\perp} \times \vec{B}) \quad \dots (24-a)$$

$$v_{\parallel}(t) = \left(\frac{qE_{\parallel}}{m} \right) t + v_{\parallel}(0) \quad \dots (24-b)$$

$$r_{\parallel}(t) = \frac{1}{2} \left(\frac{qE_{\parallel}}{m} \right) t^2 + v_{\parallel}(0)t + r_{\parallel}(0) \quad \dots (25)$$

where, $v_{\parallel}(0)$ and $r_{\parallel}(0)$ are the parallel component initial velocity and position.

To solve (24.a) it is convenient to separate $V_{\perp}$ into components

$$V_{\perp}(t) = V'_{\perp} + V_E \quad \dots (26)$$

where V_E is a constant velocity in the plane normal to B . Hence, $V'_{\perp}$ represents the particle velocity as seen by an observer in a frame of reference moving with a constant velocity V_E . Substituting equation (25) into (24) and writing the component electric field perpendicular to B in the form.

$$E_{\perp} = - \frac{(E_{\perp} \times B)}{B^2} \times B \quad (27)$$

We obtain,

$$m \frac{dV'_{\perp}}{dt} = q \left[V'_{\perp} + V_E - \frac{E_{\perp} \times B}{B^2} \right] \times B$$

If we take,

$$V_E = \frac{E_{\perp} \times B}{B^2}$$

Then,

$$m \frac{dV'_{\perp}}{dt} = q (V'_{\perp} \times B)$$

$$m \frac{dV'_{\perp}}{dt} = \frac{q}{m} (V'_{\perp} \times B)$$

It shows that if reference system moving with velocity V_E , then particle describe a circular motion at the cyclotron frequency Ω_c with radius r_c .

$$V'_{\perp} = \bar{\Omega} \times \bar{r}_c \quad \dots (29)$$

The resultant particle velocity can be expressed in vector form, independent of coordinate system as

$$V(t) = \Omega_c \times r_c + \frac{E_{\perp} \times B}{B^2} + \left(\frac{qE_{\parallel}}{m} \right) t + v_{\parallel}(0) \quad \dots (30)$$

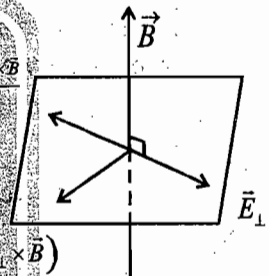
- (1) The first term represented the cyclotron circular motion.
- (2) The second term represent the drift velocity of guiding centre.
- (3) 3rd term represented the constant acceleration of guiding centre along B .
- (4) 4th term is the initial velocity component along $\bar{B}$.

Solution in Cartesian Coordinates:

Let us choose a Cartesian coordinate system with the z-axis pointing in the direction B , so that

$$\bar{B} = B\hat{z} \quad \dots (31)$$

$$E = E_x\hat{x} + E_y\hat{y} + E_z\hat{z} \quad \dots (32)$$



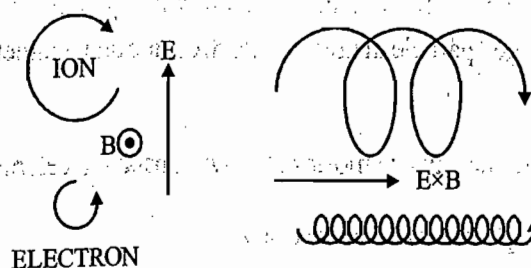


Fig. Cycloidal trajectories described by ions and electrons in crossed electric and magnetic fields. The electric field E acting together with the magnetic flux density B gives rise to a drift velocity in the direction given by $E \times B$

Using (20), the equation of motion, can be written as

$$\frac{dv}{dt} = \frac{q}{m} \left[(E_x + v_y B) \hat{x} + (E_y - v_x B) \hat{y} + E_z \hat{z} \right] \quad \dots (33)$$

As before, we consider, in what follows, a positive charge. The results for a negative charge can be obtained by changing the sign of Ω_c in the results for the positive charge.

The z component of (33) can be integrated directly and gives the same results expressed in (24-b)

and (25) for the x and y components, we first take the derivative of $\frac{dv_x}{dt}$ with respect to time and substitute

the expression for $\frac{dv_y}{dt}$, which gives

$$\frac{d^2 v_x}{dt^2} + \Omega_c^2 v_x = \Omega_c^2 \frac{E_y}{B} \quad \dots (34)$$

This is inhomogeneous equation for a harmonic oscillator of frequency Ω_c . Its solution is given by the sum of homogeneous equation solution, given in (11), with a particular solution (which is clearly given by $\frac{E_y}{B}$).

Thus,

$$v_x(t) = v'_\perp \sin(\Omega_c t + \theta_0) + \frac{E_y}{B} \quad \dots (35)$$

where $v'_\perp$ and θ_0 are integration constants. The solution for $v_y(t)$ can be obtained by substituting (35) directly into (34). Hence,

$$v_y(t) = \frac{1}{\Omega_c} \frac{dv_x}{dt} - \frac{E_x}{B} = v'_\perp \cos(\Omega_c t + \theta_0) - \frac{E_x}{B} \quad \dots (36)$$

Therefore, the velocity components $v_x(t)$ and $v_y(t)$, in the plane perpendicular to B , oscillate at the cyclotron frequency Ω_c and with amplitude $v'_\perp$. This motion is superposed to a constant drift velocity v_E given by

$$v_E = \frac{E_y}{B} \hat{x} - \frac{E_x}{B} \hat{y} \quad \dots (37)$$

One more integration of (30) and (36) gives the particle trajectory in the (x, y) plane

$$x(t) = -\frac{v'_\perp}{\Omega_c} \cos(\Omega_c t + \theta_0) + \frac{E_y}{B} t + x_0 \quad \dots (38)$$



where x_0 and y_0 are defined according to (13) and (17), but with v_T replaced by v'_T

In summary, the motion of a charged particle in uniform electrostatic and magnetostatic fields consists of three components.

(a) If E and B both are parallel to the particle motion, then a constant acceleration $\frac{qE_{\parallel}}{m}$ along the $\vec{B}$ field. If $E_{\parallel} = 0$, the particle moves along $\vec{B}$ with its initial velocity.

(b) A rotation in the plane normal to $\vec{B}$ at the cyclotron frequency $\Omega_c = |q|B/m$ and radius $r_c = \frac{v'_{\perp}}{\Omega_c}$.

(c) An electromagnetic drift velocity $v_E = (E \times B)/B^2$, perpendicular to both $\vec{B}$ and $\vec{E}$.

General formula for drift velocity due any external force

$$\vec{v} = \frac{\vec{E} \times \vec{B}}{B^2} = \frac{q\vec{E} \times \vec{B}}{qB^2} = \frac{\vec{F} \times \vec{B}}{qB^2} \Rightarrow \boxed{\vec{v} = \frac{\vec{F} \times \vec{B}}{qB^2}} \quad \dots (40)$$

This force can be gravitational, electrostatic force, any others force.

SOLVED PROBLEMS

1. A magnetic field $\vec{B} = B_0(\hat{i} + 2\hat{j} - 4\hat{k})$ exists at a point. If a test charge moving with a velocity $\vec{v} = v_0(3\hat{i} - \hat{j} + 2\hat{k})$ experiences no force at a certain point. The electric field at that point in SI units is

(a) $\vec{E} = -v_0 B_0(3\hat{i} - 2\hat{j} - 4\hat{k})$

(b) $\vec{E} = -v_0 B_0(\hat{i} + \hat{j} + 7\hat{k})$

[JEST 2012]

(c) $\vec{E} = v_0 B_0(14\hat{j} + 7\hat{k})$

(d) $\vec{E} = -v_0 B_0(14\hat{j} + 7\hat{k})$

Soln. $\vec{B} = B_0(\hat{i} + 2\hat{j} - 4\hat{k})$

$$\vec{v} = v_0(3\hat{i} - \hat{j} + 2\hat{k})$$

Lorentz force,

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B}) = 0$$

$$\Rightarrow q\vec{E} = -q(\vec{v} \times \vec{B})$$

$$\Rightarrow \vec{E} = \vec{B} \times \vec{v}$$

$$= vB_0 \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & -4 \\ 3 & -1 & 2 \end{vmatrix} = \hat{i}(0) + \hat{j}(-14) + \hat{k}(-7) = -v_0 B_0(14\hat{j} + 7\hat{k})$$

Correct option is (d)

2. The value of the magnetic field required to maintain non-relativistic protons of energy 1 MeV in a circular orbit of radius 100 mm is _____ Tesla. [GATE 2014]



Soln. $\frac{1}{2}mv^2 = 1 \times 10^6 \times 1.6 \times 10^{-19} \text{ J}$

$$\Rightarrow v = \sqrt{\frac{3.2 \times 10^{-13}}{m}} = \sqrt{\frac{3.2 \times 10^{-13}}{1.67 \times 10^{-27}}} = \sqrt{1.91677 \times 10^{14}} = 1.384474 \times 10^7 \text{ m/sec}$$

Now, for circular orbit

$$qvB = \frac{mv^2}{r}$$

$$B = \frac{mv}{qr} = \frac{1.67 \times 10^{-27} \times 1.384474 \times 10^7}{1.6 \times 10^{-19} \times 10^{-1}} = \frac{1.67 \times 1.384474}{1.6} = 1.4450 \text{ Tesla}$$

3. A proton moves with a speed of 300 m/s in a circular orbit in the xy-plane in a magnetic field 1 tesla along the positive z-direction. When an electric field of 1 V/m is applied along the positive y-direction, the centre of the circular orbit

[NET June 2015]

- (a) remains stationary
- (b) moves at 1 m/s along the negative x-direction
- (c) moves at 1 m/s along the positive z-direction
- (d) moves at 1 m/s along the positive x-direction

Soln. The drift velocity of guiding centre is given by

$$\vec{v}_E = \frac{\vec{E} \times \vec{B}}{B^2} = \frac{\hat{y} \times \hat{z}}{1} = \hat{x}$$

Therefore, the particle will move at 1 m/sec along positive x-direction.

Correct option is (d)

Four particles—electron, proton, He and Li—are projected in a circular orbit of same radius perpendicular to a given magnetic field. Then the velocity will be smallest for

- (a) electron
- (b) He⁺
- (c) proton
- (d) Li⁺

Soln. We know that radius of the circle

$$r_L = \frac{mv}{qB}$$

Since, q, B and r are fixed

$$\therefore v \propto \frac{1}{m}$$

Therefore, the velocity of He⁺ will be smallest.

Correct option is (c)

A positively charged particle, with a charge q , enters a region in which there is a uniform electric field $\vec{E}$ and a uniform magnetic field $\vec{B}$, both directed parallel to the positive y-axis. At $t = 0$, the particle is at the origin and has a speed v_0 directed along the positive x-axis. The orbit of the particle, projected on the x-z plane, is a circle. Let T be the time taken to complete one revolution of this circle. The y-coordinate of the particle at $t = T$ is given by

(a) $\frac{\pi^2 m E}{q B^2}$

(b) $\frac{2\pi^2 m E}{q B^2}$

(c) $\frac{\pi^2 m E}{q B^2} + \frac{v_0 \pi m}{q B}$

(d) $\frac{2\pi m v_0}{q B}$

Soln. Force along y direction $q\vec{E}$.

Therefore, acceleration along y-axis, $f = \frac{q\vec{E}}{m}$

For circular motion we can write

$$\frac{mv^2}{r} = qvB \Rightarrow \frac{v}{r} = \left(\frac{qB}{m} \right)$$

Time taken for complete one revolution,

$$T = \frac{2\pi r}{v} = \left(2\pi \frac{m}{qB} \right)$$

Therefore, the y-coordinate at a time $t = T$ is given by

$$y = \frac{1}{2} f T^2 = \frac{1}{2} \frac{qE}{m} \frac{4\pi^2 m^2}{q^2 B^2}$$

$$= \frac{2\pi^2 m E}{q B^2}$$

Correct option is (b)

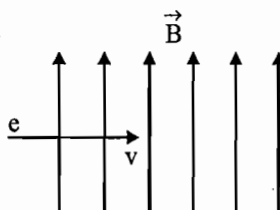


3
4
5
6
7
8



PRACTICE SET

- A charge q is revolving with angular velocity ω in a plane perpendicular to a uniform magnetic field B . If the field is reduced to $B/2$.
 (a) The linear velocity is doubled (b) The linear velocity is halved
 (c) The radius is halved (d) The radius is doubled.
- A stream of electrons enters, from left to right, a uniform magnetic field (B) which is pointing upwards as shown in the figure. The field deflects the stream of electrons.



- (a) Downward (in the plane of the paper) (b) Upward (in the plane of the paper)
 (c) Into the paper (d) Out of the Paper.
- The ratio of cyclotron frequencies of an α -particle and a proton moving in the same magnetic field $\vec{B}$ is:
 (a) $1/2$ (b) 2 (c) 4 (d) $1/4$
- An electron moves with velocity $\vec{v} = \hat{j}v_0$ under the combined influence of $\vec{E} = \hat{i}E_0$ and $\vec{B} = -\hat{k}B_0$ fields. The force on the electron is in the
 (a) Y direction (b) Z direction (c) YZ plane (d) X direction
- A charged particle is projected with certain velocity parallel to a magnetic field. It will
 (a) trace a circular path
 (b) trace a helical path
 (c) come to rest instantly
 (d) continue its motion along a straight line without any change
- Time taken by a charged particle to move once around a circular orbit in a magnetic field is independent of
 (a) mass (b) strength of magnetic field
 (c) charge (d) speed
- Four particles-electron, proton, He^+ and Li^+ are projected with the same velocity perpendicular to a given magnetic field. Then the circle described will be smallest for
 (a) electron (b) He^+ (c) proton (d) Li^+
- A uniformly charged spherical metal ball of mass M , radius R and charge q is rotating with an angular velocity ω about one of its diameter. Then the ratio of its magnetic dipole moment to the mechanical moment (angular momentum) is
 (a) $\frac{q}{2MR}$ (b) $\frac{q}{M}$ (c) $\frac{q}{2M}$ (d) $\frac{q}{M\omega}$

ANSWER KEY

Questions	1	2	3	4	5
Option	(d)	(c)	(a)	(d)	(d)
Questions	6	7	8		
Option	(d)	(d)	(c)		

Chapter 9

Magnetostatics

Introduction :

A static charge produces electric field as soon as starts to move it produces magnetic field. The field is produced by the steady current is constant with time. For this the study of magnetic effects of steady current is known as magnetostatics.

Electric current:

The rate of flow of charges with time is called current.

$$I = \frac{dq}{dt} \quad (1)$$

Current density :

The current crossing through the unit area perpendicular to direction of flow is called current density.

$$\vec{J} = \frac{dI}{ds}$$

$$I = \oint \vec{J} \cdot d\vec{s} \quad (2)$$

Also current density is given,

$$\vec{J} = nq\vec{v}$$

where n is the charge particle per unit volume, q is the charge, and v is the velocity.

Equation of Continuity:

Let us consider a volume V enclosed by a surface s . If J is the current density at any surface element ds then total current flowing through s will be

$$I = \oint \vec{J} \cdot d\vec{s}$$

It gives the rate at which charge is flowing out of the volume enclosed by the surface s . According to local conservation charge condition.

$$I = \oint \vec{J} \cdot d\vec{s} = -\frac{dq}{dt} = -\frac{d}{dt} \int \rho dV$$

$$\Rightarrow \oint \vec{J} \cdot d\vec{s} = -\frac{d}{dt} \int \rho dV$$

$$\Rightarrow \int (\vec{\nabla} \cdot \vec{J}) \cdot dV = -\int \frac{d\rho}{dt} \cdot dV$$

$$\Rightarrow \boxed{\vec{\nabla} \cdot \vec{J} + \frac{d\rho}{dt} = 0} \quad \dots (3)$$

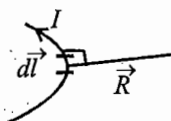
For steady current $\frac{d\rho}{dt} = 0$

$$\therefore \boxed{\vec{\nabla} \cdot \vec{J} = 0} \quad \dots (4)$$

Biot-Savart's law:

The Biot-Savart law defines the magnetic field produced by a steady current. According to this law the magnetic field at any point due to current element $I d\vec{\ell}$ is given by

$$d\vec{B} = \frac{\mu_0}{4\pi} \frac{I d\vec{\ell} \times \vec{R}}{R^3} \quad \dots (5)$$



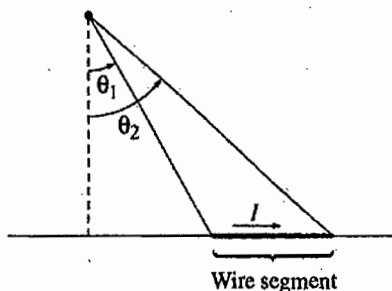
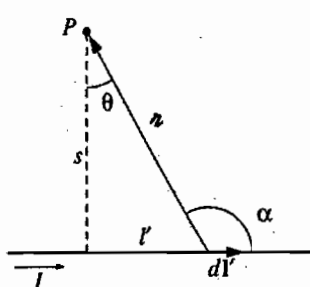
For continuous current density

$$\vec{B} = \frac{\mu_0}{4\pi} \int \frac{\vec{J} \times \vec{R}}{R^3} dV \quad \text{for volume current density} \quad \dots (6)$$

$$\vec{B} = \frac{\mu_0}{4\pi} \int \frac{\vec{k} \times \vec{R}}{R^3} ds \quad \text{for surface current density.} \quad \dots (7)$$

Application of Biot-Savart Law:

(1) Find the magnetic field a distance s from a long straight wire carrying a steady current I .



In the diagram, $(d\vec{\ell} \times \hat{r})$ points out of the page, and has the magnitude

$$d\ell' \sin \alpha = d\ell' \cos \theta$$

Also, $\ell' = s \tan \theta$, So, $d\ell' = \frac{s}{\cos^2 \theta} d\theta$

And $s = r \cos \theta$, So, $\frac{1}{r^2} = \frac{\cos^2 \theta}{s^2}$

$$\text{Thus, } B = \frac{\mu_0 I}{4\pi} \int_{\theta_1}^{\theta_2} \left(\frac{\cos^2 \theta}{s^2} \right) \left(\frac{s}{\cos^2 \theta} \right) \cos \theta d\theta$$



In the case of infinite wire, $\theta_1 = -\pi/2$ and $\theta_2 = \pi/2$, so we obtain

$$B = \frac{\mu_0 I}{2\pi s} \quad \dots (8)$$

The field circles around the wire, following right hand thumb rule.

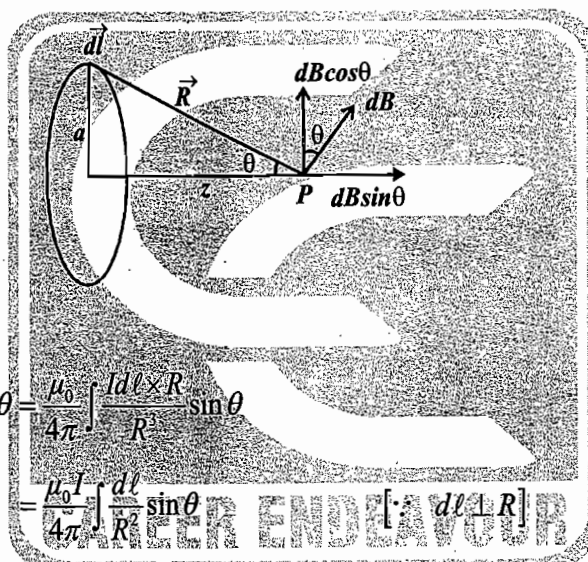
(2) A circular loop of radius carrying current I . Find the magnetic field on the axis at distance from the centre of the loop.

The field at the point P due to small element $d\ell$ is given by

$$d\vec{B} = \frac{\mu_0}{4\pi} \frac{Id\vec{\ell} \times \vec{R}}{R^3}$$

It has two components $dB \sin \theta$ and $dB \cos \theta$

The component $dB \cos \theta$ will be cancelled out due to symmetry.



Therefore, the field at P is

$$\begin{aligned} B &= \int dB \sin \theta = \frac{\mu_0}{4\pi} \int \frac{Id\ell \times R}{R^3} \sin \theta \\ &= \frac{\mu_0 I}{4\pi} \int \frac{d\ell}{R^2} \sin \theta \quad [\because d\ell \perp R] \\ &= \frac{\mu_0 I}{4\pi} \int \frac{d\ell}{(z^2 + a^2)} \frac{a}{\sqrt{z^2 + a^2}} \\ &= \frac{\mu_0 I}{4\pi} \frac{2\pi a^2}{(z^2 + a^2)^{3/2}} \hat{z} = \frac{\mu_0 I}{2} \frac{a^2}{(z^2 + a^2)^{3/2}} \hat{z} \quad \dots (9) \end{aligned}$$

At the centre of circular loop is given by

$$\vec{B} = \frac{\mu_0 I}{2a} \hat{z} \quad \dots (10)$$

(3) Helmholtz coil:

Let us consider two such coil (1) and (2) each of radius a , same number of turn and same current. Separated by the distance $2d$. Let consider a point P at a small distance x from the mid-point

$$\vec{B} = \frac{\mu_0 IN}{2} \left[\frac{a^2}{\{a^2 + (d+x)^2\}^{3/2}} + \frac{a^2}{\{a^2 + (d-x)^2\}^{3/2}} \right]$$

$$\Rightarrow \frac{d+x}{\{a^2 + (d+x)^2\}^{5/2}} = \frac{d-x}{\{a^2 + (d-x)^2\}^{5/2}}$$

$$\frac{d+x}{d-x} = \left[\frac{a^2 + (d+x)^2}{a^2 + (d-x)^2} \right]^{5/2}$$

$$\Rightarrow \frac{d+x}{d-x} = \frac{1 + \frac{5dx}{d^2 + a^2}}{1 - \frac{5dx}{d^2 + a^2}} \Rightarrow \frac{x}{d} = \frac{5dx}{d^2 + a^2}$$

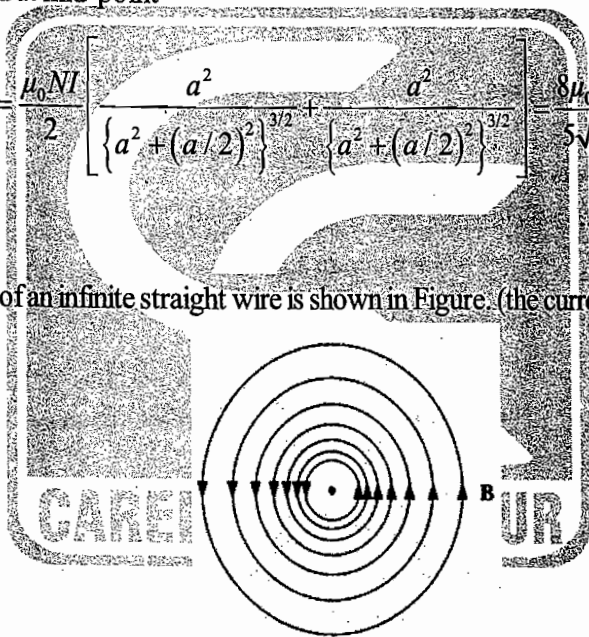
$$\Rightarrow a^2 = 4d^2 \quad \Rightarrow 2d = a$$

The magnetic field near the mid-point will be uniform if the separation between them equal to the radius. Therefore, the field at mid-point

$$B = \frac{\mu_0 NI}{2} \left[\frac{a^2}{\{a^2 + (a/2)^2\}^{3/2}} + \frac{a^2}{\{a^2 + (a/2)^2\}^{3/2}} \right] = \frac{8\mu_0 NI}{5\sqrt{5}a} \quad \dots (11)$$

Ampere's law:

The magnetic field of an infinite straight wire is shown in Figure. (the current is coming out of the page).



According to equation (10), the integral of B around a circular path of radius s , centered at the wire is:

$$\oint B \cdot d\ell = \oint \frac{\mu_0 I}{2\pi s} d\ell = \frac{\mu_0 I}{2\pi s} \oint d\ell \quad \dots (12)$$

Notice that the answer is independent of s . In fact, it doesn't have to be a circle; any old loop that encloses the wire would give the same answer. For if we use cylindrical coordinates (s, ϕ, z) , with the current flowing along the z -axis; we get

$$B = \frac{\mu_0 I}{2\pi s} \hat{\phi}$$

And, $d\ell = ds \hat{s} + s d\phi \hat{\phi} + dz \hat{z}$, So

$$\oint B \cdot d\ell = \frac{\mu_0 I}{2\pi} \int_0^{2\pi} \frac{1}{s} s d\phi = \frac{\mu_0 I}{2\pi} \int_0^{2\pi} d\phi = \mu_0 I$$

This equation is true not only for a straight wire, but also for any current distribution. This will basically become our Ampere's law if we just replace the current I by enclosed current I_{enc} .



$$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{\text{enc}} \quad \dots (13)$$

If the flow of charge is represented by a volume current density $\mathbf{J}$, then from the definition of $\mathbf{J}$ we get

$$\oint \mathbf{B} \cdot d\mathbf{l} = \int (\nabla \times \mathbf{B}) \cdot d\mathbf{a} = \mu_0 \int \mathbf{J} \cdot d\mathbf{a}$$

Hence, $\nabla \times \mathbf{B} = \mu_0 \mathbf{J} \quad \dots (14)$

That's the curl of $\mathbf{B}$. Remember the result. It will be needed many times.

Divergence of magnetic field is derivable both from Ampere's law and Biot Savart Law. The result is

$$\nabla \cdot \mathbf{B} = 0 \quad \dots (15)$$

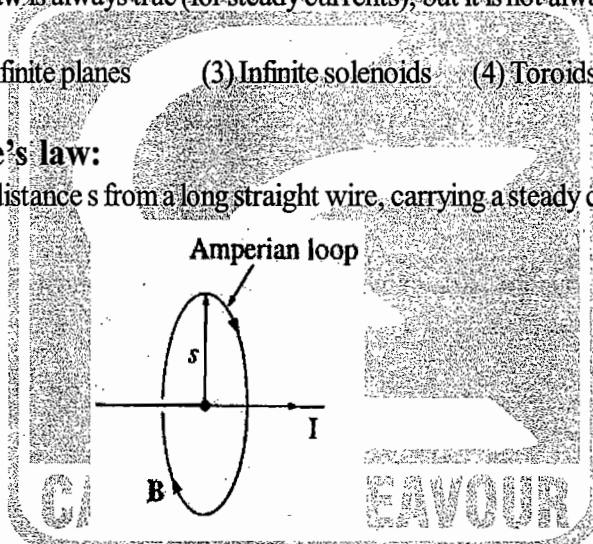
What is the significance of these two results. Well, finite value of curl of magnetic field (proportional to current) simply means that the Magnetic field always curl around the current. And the zero divergence means the magnetic field, unlike the electric field, does not emerge and solely "diverge" outside from a single point. If magnetic field line has started from somewhere, it must end somewhere. In other words there are no magnetic monopoles. Notice the different nature of electric and magnetic field.

Like Gauss's law, Ampere's law is always true (for steady currents), but it is not always useful. It is only useful in case of—

- (1) Infinite straight lines (2) Infinite planes (3) Infinite solenoids (4) Toroids

Application of Ampere's law:

- (1) Find the magnetic field a distance s from a long straight wire, carrying a steady current I .



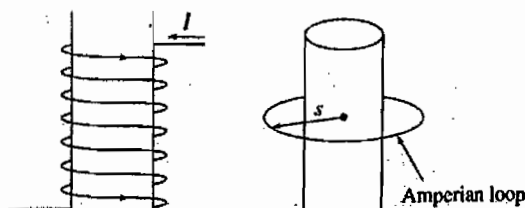
We know the direction of $\mathbf{B}$ is "circumferential," circling around the wire as indicated by the right hand thumb rule. By symmetry, the magnitude of $\mathbf{B}$ is constant around an Amperian loop of radius s , centered on the wire. So Ampere's law gives

$$\oint \mathbf{B} \cdot d\mathbf{l} = B \oint d\mathbf{l} = B 2\pi s = \mu_0 I_{\text{enc}} = \mu_0 I$$

So, we get, $B = \frac{\mu_0 I}{2\pi s}$

We get same answer before.

- (2) Find the magnetic field of a very long solenoid, consisting of n closely wound turns per unit length on a cylinder of radius R and carrying a steady current I . The windings are so close that one can take each wind circular.

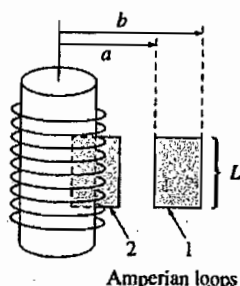


It shall be clear that the magnetic field $\mathbf{B}$ will not have any radial or circumferential components. If it had radial components than changing the direction of current could alter the direction of field. But changing the direction of current just means the upside downing of the solenoid. So the radial field won't be there.

For B_ϕ would be constant around an amperian loop concentric with the solenoid. Hence

$$\oint \mathbf{B} \cdot d\mathbf{l} = B_\phi (2\pi s) = \mu_0 I_{\text{enc}} = 0$$

So the magnetic field of an infinite, closely wound solenoid runs parallel to the axis. From the right hand rule, we expect that it points upward inside the solenoid and downward outside. Apply Ampere's law to the two rectangular loops in Fig. 5.19. Loop 1 lies entirely outside the solenoid, with its sides at distances 'a' and 'b' from the axis:



$$\oint \mathbf{B} \cdot d\mathbf{l} = [B(a) - B(b)]L = \mu_0 I_{\text{enc}} = 0$$

So, $B(a) = B(b)$.

Evidently the field outside does not depend on the distance from the axis. But we know that it goes to zero for large s . It must therefore be zero everywhere! So outside the solenoid the field is zero.

As for loop 2, which is half inside and half outside, Ampere's law gives

$$\oint \mathbf{B} \cdot d\mathbf{l} = BL = \mu_0 I_{\text{enc}} = \mu_0 n I L$$

Where B is the field inside the solenoid.

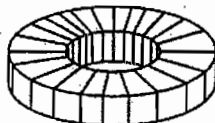
So the field is

$$B = \begin{cases} \mu_0 n I & \text{inside the solenoid} \\ 0 & \text{outside the solenoid} \end{cases} \quad \dots (16)$$

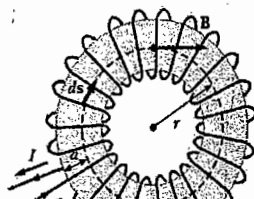
Notice that the field inside is uniform.

(3) Field of a Toroid

A device called a toroid is often used to create an almost uniform magnetic field in some enclosed area. The device consists of a conducting wire wrapped around a ring (a torus) made of a non-conducting material. For a toroid having N closely spaced turns of wire, calculate the magnetic field in the region occupied by the torus, a distance r from the center.



To calculate this field, we must evaluate $\oint \mathbf{B} \cdot d\mathbf{s}$ over the circle of radius r . By symmetry, we see that the magnitude of the field is constant on this circle and tangent to it, so Furthermore, note that





the circular closed path surrounds N loops of wire, each of which carries a current I . Therefore, the right side of Ampere's law is $\mu_0 NI$ in this case.

Ampère's law applied to the circle gives $\oint \vec{B} \cdot d\vec{s} = B \oint ds = B(2\pi r) = \mu_0 NI$

$$B = \frac{\mu_0 NI}{2\pi r} \quad \dots (17.a)$$

This result shows that B varies as $1/r$ and hence is non-uniform in the region occupied by the torus. However, if r is very large compared with the cross-sectional radius of the torus, then the field is approximately uniform inside the torus.

Infinite sheet of current:

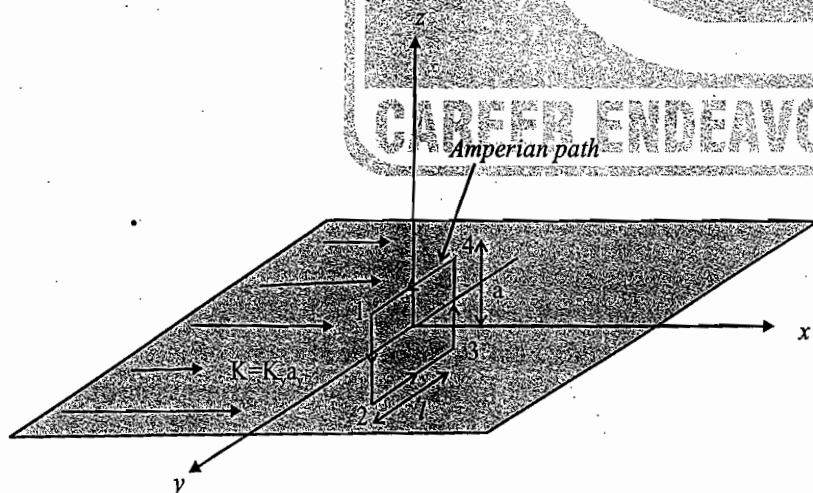
Consider an infinite plane $z=0$ on which a uniform surface current density $\vec{K} = K\hat{i}$ flows shown in the figure. Dividing the current sheet into pairs of current filaments at $+y$ and $-y$ it is easy to show the resultant magnetic field $\vec{B}$ will be directed along $-y$ -axis for $z > 0$ and along $+y$ axis for $z < 0$. With this in mind we choose a rectangle PQRS parallel to yz -plane as an Amperian loop. Now, applying Ampere's law we get,

$$\oint \vec{B} \cdot d\vec{\ell} = \mu_0 K \ell$$

$$\text{Or,} \quad \int_{23} \vec{B} \cdot d\vec{\ell} + \int_{34} \vec{B} \cdot d\vec{\ell} + \int_{41} \vec{B} \cdot d\vec{\ell} + \int_{12} \vec{B} \cdot d\vec{\ell} = \mu_0 K \ell$$

$$\text{Or,} \quad B\ell + 0 + B\ell + 0 = \mu_0 K \ell \quad \text{or,} \quad B = \frac{\mu_0}{2} K$$

Note that B is independent of the distance from the current sheet.



Vectorially,

$$\vec{B} = \begin{cases} -\frac{\mu_0 K}{2} \hat{j} & \text{for } z > 0 \\ +\frac{\mu_0 K}{2} \hat{j} & \text{for } z < 0 \end{cases} \quad \dots (17.b)$$

$$\vec{B} = \frac{\mu_0}{2} K \hat{j} \quad \dots (17.c)$$



Magnetic dipole

You remember what is an electric dipole, right? When two opposite charge placed a small distance apart, it called an electric dipole. Similarly, when a wire carrying a current forms a small closed loop, then it is called magnetic dipole.

If $\mathbf{m}$ is the magnetic moment of magnetic dipole then

$$\vec{m} \equiv I \oint d\mathbf{a} = I\mathbf{a} \quad \dots (18)$$

Here $\mathbf{a}$ is the "vector area" of the loop. If the loop is flat, $\mathbf{a}$ is the ordinary area enclosed, with the direction assigned by the usual right hand rule (fingers in the direction of the current).

Magnetic vector potential of a magnetic dipole with magnetic moment $\mathbf{m}$ is:

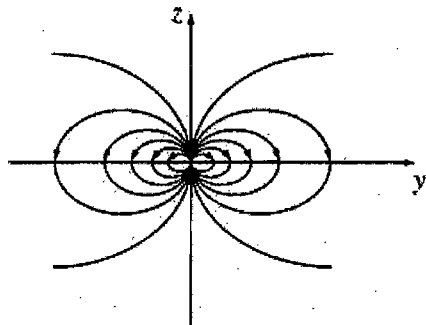
$$\vec{A}_{\text{dip}}(\mathbf{r}) = \frac{\mu_0}{4\pi} \frac{\vec{m} \times \hat{\mathbf{r}}}{r^2} \quad \dots (19)$$

In polar coordinate $\mathbf{A}$ becomes

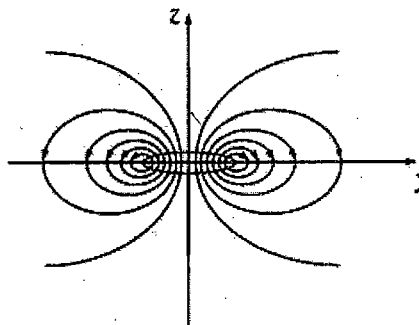
$$\vec{A}_{\text{dip}}(\mathbf{r}) = \frac{\mu_0}{4\pi} \frac{m \sin \theta}{r^2} \hat{\phi} \quad \dots (20)$$

Hence the magnetic field is:

$$\vec{B}_{\text{dip}}(\mathbf{r}) = \vec{\nabla} \times \vec{A} = \frac{\mu_0 m}{4\pi r^3} (2 \cos \theta \hat{\mathbf{r}} + \sin \theta \hat{\theta}) \quad \dots (21)$$



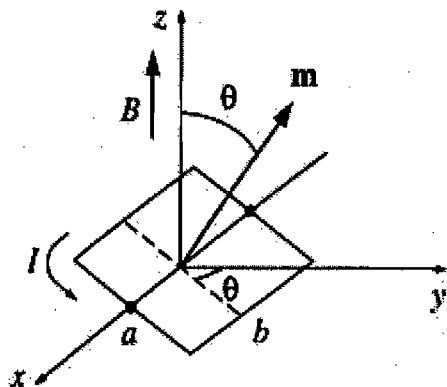
(a) Field of a "pure" dipole



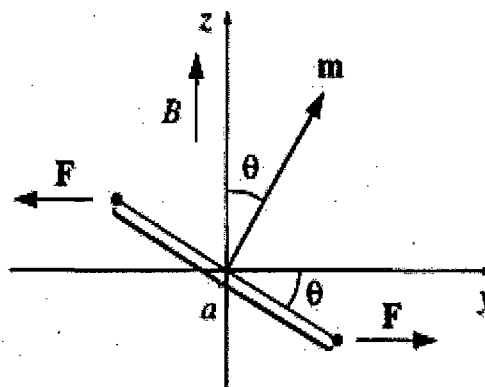
(a) Field of a "physical" dipole

Torque on a magnetic dipole

A magnetic dipole experiences a torque in a magnetic field, just as an electric dipole does in an electric field. Let's calculate the torque on a rectangular current loop in a uniform field $\mathbf{B}$.



(a)



(b)



Take the center of loop at the origin, and tilt it an angle θ from the z axis towards the y -axis. Let B point in the z -direction. The forces on the two sloping sides cancel as they tend to stretch the loop, but they don't rotate it. The forces on the "horizontal" sides are likewise equal and opposite (so the net force on the loop is zero), but they do generate a torque:

$$\vec{N} = aF \sin \theta \hat{x}$$

The magnitude of the force on each of these segments is

$$\vec{F} = I b B$$

And hence the torque is

$$\vec{N} = I ab B \sin \theta \hat{x} = mB \sin \theta \hat{x}$$

Or,

$$\boxed{\vec{N} = \vec{m} \times \vec{B}} \quad \dots (22)$$

where, $m = Iab$ is the magnetic dipole moment of the loop. Equation (22) gives the exact torque on any localized current distribution, in the presence of a uniform field; in a nonuniform field it is the exact torque (about the center) for a perfect dipole of infinitesimal size.

Energy of a magnetic dipole is

$$U = -\vec{m} \cdot \vec{B} \quad \dots (23)$$

The force on a magnetic dipole for an infinitesimal loop, with dipole moment $\vec{m}$, in a field $\vec{B}$, is

$$\boxed{\vec{F} = \vec{\nabla} (\vec{m} \cdot \vec{B})} \quad \dots (24)$$

Magnetic vector potential

We have seen that

$$\vec{\nabla} \cdot \vec{B} = 0 \quad \dots (25)$$

This implies that there exist a vector function $\vec{A}$ such that

$$\boxed{\vec{B} = \vec{\nabla} \times \vec{A}} \quad \text{(From vector calculus)} \quad \dots (26)$$

This $\vec{A}$ is called the magnetic vector potential.

Calculation of $\vec{A}$ in a few simple cases:

(1) A straight current carrying conductor:

Let us consider a long straight wire AB carrying a steady current I . We are to calculate the magnetic vector potential $\vec{A}$ at any point P . We choose cylindrical coordinates (r, θ, z) with z -axis along the wire and the foot O of the perpendicular from P on the wire as the origin. Let us consider an element $d\vec{\ell} = \hat{z}dz$ at a distance z from O . Now by definition

$$\begin{aligned} \vec{A} &= \frac{\mu_0}{4\pi} \int \frac{I d\vec{\ell}}{R} = \hat{z} \frac{\mu_0 I}{4\pi} \int_{-\ell_1}^{+\ell_2} \frac{dz}{\sqrt{r^2 + z^2}} = \hat{z} \frac{\mu_0 I}{4\pi} \left[\ln \left(z + \sqrt{r^2 + z^2} \right) \right]_{-\ell_1}^{+\ell_2} \\ &= \hat{z} \frac{\mu_0 I}{4\pi} \ln \left[\frac{\ell_2 + \sqrt{r^2 + \ell_2^2}}{-\ell_1 + \sqrt{r^2 + \ell_1^2}} \right] \end{aligned}$$

If the wire is of length $2L$ and the point P is just above the centre of the wire then $\ell_1 = \ell_2 = L$ and we can



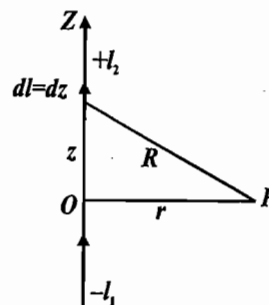
$$\vec{A} = \hat{z} \frac{\mu_0 I}{4\pi} \ln \left[\frac{L + \sqrt{r^2 + L^2}}{-L + \sqrt{r^2 + L^2}} \right]$$

Now if $r/L \ll 1$ then we can approximate

$$\vec{A} = \hat{z} \frac{\mu_0 I}{4\pi} \ln \left[\frac{1 + (1 + r^2/L^2)^{1/2}}{-1 + (1 + r^2/L^2)^{1/2}} \right] \approx \hat{z} \frac{\mu_0 I}{4\pi} \ln \left[\frac{2 + \frac{r^2}{2L^2}}{\frac{r^2}{2L^2}} \right]$$

$$= \hat{z} \frac{\mu_0 I}{4\pi} \ln \left(\frac{4L^2}{r^2} + 1 \right) \approx \hat{z} \frac{\mu_0 I}{2\pi} \ln \left(\frac{2L}{r} \right)$$

... (27)



Using equation (26), we can calculate the magnetic field due to a long straight wire as follows:

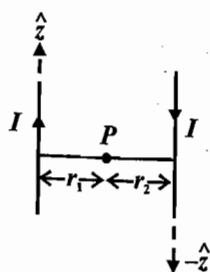
$$\begin{aligned} \vec{B} &= \vec{\nabla} \times \vec{A} = \frac{1}{r} \begin{vmatrix} \hat{r} & r\hat{\theta} & \hat{z} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ 0 & 0 & A_z \end{vmatrix} = -\hat{\theta} \frac{\partial A_z}{\partial r} \\ &= -\hat{\theta} \frac{\partial}{\partial r} \left[\frac{\mu_0 I}{2\pi} \ln \left(\frac{2L}{r} \right) \right] = \hat{\theta} \frac{\mu_0 I}{2\pi r} \end{aligned}$$

Two long straight parallel wires carrying same current:

Suppose we have two long parallel wires each of length $2L$ carrying current I in opposite directions. The magnetic vector potential $\vec{A}$ at any point P due to this system can be calculated by using the result (27). Thus,

$$\vec{A} = \vec{A}_1 + \vec{A}_2 = \hat{z} \frac{\mu_0 I}{2\pi} \ln \left(\frac{2L}{r_1} \right) - \hat{z} \frac{\mu_0 I}{2\pi} \ln \left(\frac{2L}{r_2} \right) = \hat{z} \frac{\mu_0 I}{2\pi} \ln \left(\frac{r_2}{r_1} \right) \quad \dots (28)$$

where r_1 and r_2 are the distances of the point of observation P from the wires and $\hat{z}$ is a unit vector parallel to the wires as shown in figure below.



Infinite sheet of current:

Consider which shows an infinite current sheet. From equation (17.b) we can write

$$\vec{B} = \vec{\nabla} \times \vec{A} = \pm \frac{\mu_0 K}{2} \hat{j}$$

where + and - signs apply to $z < 0$ and $z > 0$ regions, respectively. As the current is along x-axis we

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & 0 & 0 \end{vmatrix} = \pm \frac{\mu_0 K}{2} = \pm \frac{\mu_0 K}{2}$$

Integrating we get,

$$A_x = \pm \frac{\mu_0 K z}{2} + \text{constant} \quad \dots (29)$$

where the constant may be used to define the position at which $A = 0$

Long solenoid:

Consider an infinite solenoid with n turns per unit length, radius a and carrying current I . Symmetry of the problem suggests that $\vec{A}$ should have the direction of the current, i.e. the circumferential direction ($\hat{\theta}$). The magnetic field inside is along the axis (z -axis) of the solenoid and is given by

$$\vec{B} = \mu_0 n I \hat{z}$$

Now, considering a circular Amperian loop of radius r ($r < a$) inside the solenoid in xy -plane we can write.

$$\oint_C \vec{A} \cdot d\vec{\ell} = \int_S \vec{\nabla} \times \vec{A} \cdot d\vec{S} = \int_S \vec{B} \cdot d\vec{S}$$

$$\text{Or,} \quad A \cdot 2\pi r = \mu_0 n I \cdot \pi r^2$$

$$\text{Or,} \quad \vec{A} = \frac{\mu_0 n I}{2} r \hat{\theta}, \text{ for } r < a \quad (30)$$

For a circular Amperian loop of radius r ($r > a$) outside the solenoid,

$$\int_S \vec{B} \cdot d\vec{S} = \mu_0 n I \cdot \pi a^2$$

Since the field extends only up to $r = a$. Thus,

$$A \cdot 2\pi r = \mu_0 n I \cdot \pi a^2 \text{ or } \vec{A} = \frac{\mu_0 n I a^2}{2r} \hat{\theta} \text{ for } r > a$$

It is easy to show that $\vec{A}$ as given by the equation (29) or (30) satisfies the relations $\vec{\nabla} \times \vec{A} = \vec{B}$ and $\vec{\nabla} \cdot \vec{A} = 0$

$$\vec{A} = \frac{\mu_0 n I}{2} r (-\sin \theta \hat{i} + \cos \theta \hat{j}) = \frac{\mu_0 n I}{2} (-y \hat{i} + x \hat{j})$$

$$\text{Now,} \quad \vec{\nabla} \times \vec{A} = \frac{\mu_0 n I}{2} \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y & x & 0 \end{vmatrix} = \hat{k} \mu_0 n I = \vec{B} \quad (\because \hat{z} = \hat{k})$$

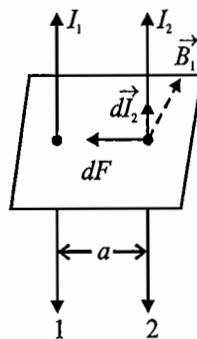
$$= \mu_0 n I \left[\frac{\partial}{\partial x} \dots \right]$$



Force between two long straight parallel conductors carrying current:

Let us consider two long parallel wires (1 and 2) carrying currents I_1 and I_2 , separated by a distance a as shown in figure. The conductor 1 produces a magnetic field

$$B_1 = \frac{\mu_0}{4\pi} \cdot \frac{2I_1}{a}$$



at the position of conductor 2. It is directed perpendicular to the conductor 2 as shown in above figure.

The $d\vec{F}$ acting on the element $d\vec{\ell}_2$ of the conductor 2 due to $\vec{B}_1$ is

$$d\vec{F} = I_2 (d\vec{\ell}_2 \times \vec{B}_1) \text{ or } |d\vec{F}| = dF = \frac{\mu_0}{4\pi} \cdot \frac{2I_1 I_2}{a} d\ell_2$$

This force is directed towards the conductor 1, i.e. attractive if I_1 and I_2 flow in the same direction. On the other hand, the force is directed away from the conductor 1 if I_1 and I_2 flow in opposite directions.

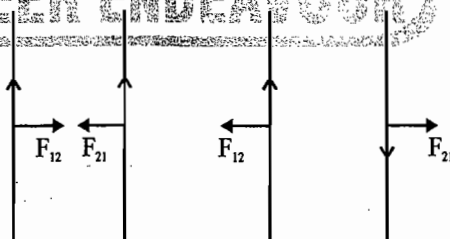
The force per unit length of each wire is of magnitude

$$|\vec{F}| = \frac{\mu_0}{4\pi} \cdot \frac{2I_1 I_2}{a} \quad \dots (31)$$

and directed perpendicularly to their lengths.

(i) If current flow same direction the force between two line is attractive in nature.

(ii) If the current flow opposite direction with each other then the force is repulsive in nature.



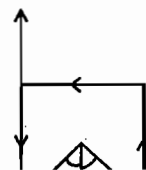
SOLVED PROBLEMS

1. A current I flows in the anticlockwise direction through a square loop of side a lying in the xoy plane with its center at the origin. The magnetic induction at the center of the square loop is : [GATE 2001]

- (a) $\frac{2\sqrt{2}\mu_0 I}{\pi a} \hat{e}_x$ (b) $\frac{2\sqrt{2}\mu_0 I}{\pi a} \hat{e}_z$ (c) $\frac{2\sqrt{2}\mu_0 I}{\pi a^2} \hat{e}_z$ (d) $\frac{2\sqrt{2}\mu_0 I}{\pi a^2} \hat{e}_x$

Soln. Magnetic field at distance $\frac{a}{2}$ from finite length wire is given by

$$B = \frac{\mu_0 I}{4\pi} (\sin \theta_1 + \sin \theta_2)$$



Since, $\theta_1 = \theta_2 \therefore B_1 = \frac{\mu_0 I}{\pi a} \sin \theta = \frac{\mu_0 I}{\pi a} \frac{1}{\sqrt{2}}$

Therefore, the total magnetic field at the centre of square is given by

$$B = B_1 + B_2 + B_3 + B_4 = 4B_1 = \frac{4\mu_0 I}{\pi a} \frac{1}{\sqrt{2}} = \frac{2\sqrt{2}\mu_0 I}{\pi a}$$

Correct option is (b)

2. Consider an infinitely long straight cylindrical conductor of radius R with its axis along the z -direction, which carries a current of 1 A uniformly distributed over its cross section. Which of the following statements is correct?

[GATE 2002]

- (a) $\vec{\nabla} \times \vec{B} = 0$ everywhere (b) $\vec{\nabla} \times \vec{B} = \frac{\mu_0}{\pi R^2} \hat{z}$ everywhere
(c) $\vec{\nabla} \times \vec{B} = 0$ for $r > R$ (d) $\vec{\nabla} \times \vec{B} = \frac{\mu_0}{\pi R^2} \hat{z}$ for $r > R$

Soln. We can write from Ampere law,

$$B = \frac{\mu_0 I}{2\pi r} \hat{\theta} \quad r > R$$

$$= \frac{\mu_0 I r}{2\pi R^2} \hat{\theta} \quad r < R$$

$$\therefore \vec{\nabla} \times \vec{B} = \frac{1}{r} \frac{\partial}{\partial r} \left(\frac{\mu_0 I}{2\pi} \right) = 0 \quad r > R$$

$$\vec{\nabla} \times \vec{B} = \frac{1}{r} \frac{\partial}{\partial r} \left(\frac{\mu_0 I r^2}{2\pi R} \right) \hat{z}$$

$$= \frac{\mu_0 I}{\pi R^2} \hat{z} \neq 0 \quad r < R$$

Correct option is (c)

3. Which one of the following current densities, $\vec{J}$, can generate the magnetic vector potential $\vec{A} = (y^2 \hat{i} + x^2 \hat{j})$?

[GATE 2009]

- (a) $\frac{2}{\mu_0} (x\hat{i} + y\hat{j})$ (b) $-\frac{2}{\mu_0} (\hat{i} + \hat{j})$ (c) $\frac{2}{\mu_0} (\hat{i} - \hat{j})$ (d) $\frac{2}{\mu_0} (x\hat{i} - y\hat{j})$

Soln. We know that,

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$$

$$\vec{\nabla} \times \vec{A} = \vec{B}$$

$$\therefore \vec{J} = \frac{1}{\mu_0} (\vec{\nabla} \times (\vec{\nabla} \times \vec{A})) = \frac{1}{\mu_0} (\vec{\nabla} \times (2x - 2y) \hat{k})$$

$$= \frac{1}{\mu_0} 2(-\hat{i} - \hat{j}) = -\frac{2}{\mu_0} (\hat{i} + \hat{j})$$

Correct option is (b)



4. The vector potential in a region is given as $\vec{A}(x, y, z) = -y\hat{i} + 2x\hat{j}$. The associated magnetic induction is $\vec{B}$ is : [GATE 2006]

- (a) $\hat{i} + \hat{k}$ (b) $3\hat{k}$ (c) $-\hat{i} + 2\hat{j}$ (d) $-\hat{i} + \hat{j} + \hat{k}$

Soln. $\vec{A} = -y\hat{i} + 2x\hat{j}$

$$\vec{B} = \vec{\nabla} \times \vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y & 2x & 0 \end{vmatrix} = (2+1)\hat{k} = 3\hat{k}$$

Correct option is (b)

5. The value of $\oint_C \vec{A} \cdot d\vec{\ell}$ along a square loop of side L in a uniform field $\vec{A}$ is : [GATE 2006]

- (a) 0 (b) $2LA$ (c) $4LA$ (d) L^2A

Soln. $\int \vec{A} \cdot d\vec{\ell} = \int (\vec{\nabla} \times \vec{A}) \cdot d\vec{s} = 0$ ($\vec{A}$ is uniform)

Correct option is (a)

6. Which of the following expressions for a vector potential $\vec{A}$ DOES NOT represent a uniform magnetic field of magnitude B_0 along the z-direction? [GATE 2011]

- (a) $\vec{A} = (0, B_0x, 0)$ (b) $\vec{A} = (-B_0y, 0, 0)$ (c) $\vec{A} = \left(\frac{B_0x}{2}, \frac{B_0y}{2}, 0\right)$ (d) $\vec{A} = \left(-\frac{B_0y}{2}, \frac{B_0x}{2}, 0\right)$

Soln. We know that, $\vec{\nabla} \times \vec{A} = \vec{B}$

$$\therefore \vec{\nabla} \times (B_0x\hat{j}) = B_0\hat{k} \text{ is uniform field.}$$

$$\vec{\nabla} \times (-B_0y\hat{j}) = -B_0\hat{k} \text{ is uniform field.}$$

$$\vec{\nabla} \times \left(\frac{B_0x}{2}\hat{i} + \frac{B_0y}{2}\hat{j}\right) = 0$$

$$\vec{\nabla} \times \left(-\frac{B_0y}{2}\hat{i} + \frac{B_0x}{2}\hat{j}\right) = \frac{B_0}{2}(\hat{k}) \text{ is uniform field.}$$

Correct option is (c)

7. A long cylindrical kept along z-axis carries a current density $\hat{J} = J_0r\hat{k}$, where J_0 is a constant and r is the radial distance from the axis of the cylinder. The magnetic induction $\vec{B}$ inside the conductor at a distance d from the axis of the cylinder is : [GATE 2006]

- (a) $\mu_0 J_0 \hat{\phi}$ (b) $\frac{\mu_0 J_0 d}{2} \hat{\phi}$ (c) $\frac{\mu_0 J_0 d^2}{3} \hat{\phi}$ (d) $\frac{\mu_0 J_0 d^3}{4} \hat{\phi}$

Soln. We know that, Ampere law,

$$\oint \vec{B} \cdot d\vec{\ell} = \mu_0 \int \vec{J} \cdot d\vec{s}$$



$$\Rightarrow B \cdot 2\pi d = \mu_0 J_0 \int_0^d \int_0^{2\pi} r dr d\theta$$

$$\Rightarrow B \cdot 2\pi d = \mu_0 J_0 \frac{d^3}{3} 2\pi$$

$$\Rightarrow B = \frac{\mu_0 J_0 d^2}{3} \hat{\phi}$$

Correct option is (c)

Statement for linked Answer Q.8 and Q.9:

An infinitely long hollow cylinder of radius R carrying a surface charge density s is rotated about its cylindrical axis with a constant angular speed ω

8. The magnitude of the surface current is :

[GATE 2005]

- (a) $\sigma R \omega$ (b) $2\sigma R \omega$ (c) $\pi \sigma R \omega$ (d) $2\pi \sigma R \omega$

Soln. We know that current,

$$I = \frac{Q}{T} = \frac{2\pi R \sigma}{T} = \omega R \sigma$$

Correct option is (a)

9. The magnitude of vector potential inside the cylinder at a distance from its axis is :

- (a) $2\mu_0 \sigma R \omega r$ (b) $\mu_0 \sigma R \omega r$ (c) $\frac{1}{2} \mu_0 \sigma R \omega r$ (d) $\frac{1}{4} \mu_0 \sigma R \omega r$ [GATE 2005]

Soln. We know that,

$$\oint \vec{A} \cdot d\vec{\ell} = \int \vec{B} \cdot d\vec{s} \Rightarrow A \cdot 2\pi r = \mu_0 \sigma \omega R \cdot \pi r^2$$

$$\Rightarrow A = \frac{1}{2} \mu_0 \sigma \omega R r$$

Correct option is (c)

10. A toroidal coil has N closely-wound turns. Assume the current through the coil to be I and the toroid is filled with a magnetic material of relative permittivity μ_r . The magnitude of magnetic induction $\vec{B}$ inside the toroid, at a radial distance r from the axis, is given by [GATE 2007]

- (a) $\mu_r \mu_0 N I r$ (b) $\frac{\mu_r \mu_0 N I}{r}$ (c) $\frac{\mu_r \mu_0 N I}{2\pi r}$ (d) $2\pi \mu_r \mu_0 N I r$

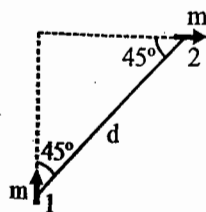
Soln. $\oint \vec{B} \cdot d\vec{\ell} = \mu_r \mu_0 I N$

$$\Rightarrow B = \frac{\mu_r \mu_0 I N}{2\pi r}$$

Correct option is (c)



11. Two magnetic dipoles of magnitude m each are placed in a plane as shown



The energy of interaction is given by

[GATE 2010]

- (a) Zero (b) $\frac{\mu_0 m^2}{4\pi d^3}$ (c) $\frac{3\mu_0 m^2}{2\pi d^3}$ (d) $\frac{3\mu_0 m^2}{8\pi d^3}$

Soln. The interaction energy between two magnetic dipole moment is given by

$$w_{12} = \frac{\mu_0}{4\pi} \left[3 \frac{(\vec{m}_1 \cdot \vec{r}_{12}) \vec{r}_{12}}{r_{12}^5} - \frac{\vec{m}_1}{r_{12}^3} \right] \cdot \vec{m}_2$$

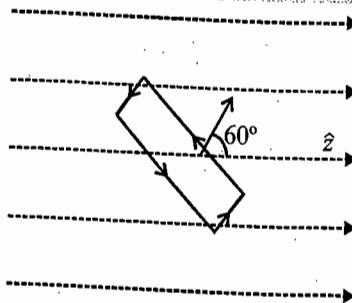
$$= \frac{\mu_0}{4\pi} \left[3 \frac{(\vec{m}_1 \cdot \vec{r}_{12})(\vec{m}_2 \cdot \vec{r}_{12})}{r_{12}^5} - \frac{\vec{m}_1 \cdot \vec{m}_2}{r_{12}^3} \right]$$

$$= \frac{\mu_0}{4\pi} \left[\frac{3m^2 \cos^2 45^\circ}{d^3} - 0 \right] \quad (\because \vec{m}_1 \perp \vec{m}_2)$$

$$= \frac{\mu_0}{4\pi} \frac{3m^2}{d^3} \frac{1}{2} = \frac{\mu_0}{8\pi} \frac{3m^2}{d^3}$$

Correct option is (d)

12. In a constant magnetic field of 0.6 Tesla along the z direction, find the value of the path integral $\oint \vec{A} \cdot d\vec{l}$ in the units of (Tesla m^2) on a square loop of side length $(1/\sqrt{2})$ meters. The normal to the loop makes an angle of 60° to the z -axis, as shown in the figure. The answer should be up to two decimal places. _____



[GATE 2013]

Soln. $\oint \vec{A} \cdot d\vec{l} = \oint (\vec{\nabla} \times \vec{A}) \cdot d\vec{S} = \oint \vec{B} \cdot d\vec{S} = BA \cos \theta = 0.6 \times \frac{1}{2} \cdot \cos 60^\circ = 0.6 \times \frac{1}{2} \times \frac{1}{2} = 0.15$

13. If the vector potential

[GATE 2014]

$$\vec{A} = \alpha x \hat{x} + 2y \hat{y} - 3z \hat{z}$$

satisfies the Coulomb gauge, the value of the constant α is _____

Soln. The condition of Coulomb gauge

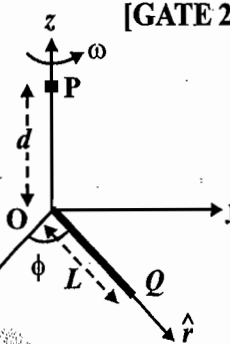
$$\vec{\nabla} \cdot \vec{A} = 0$$

$$\Rightarrow \frac{\partial}{\partial x}(\alpha x) + \frac{\partial}{\partial y}(2y) + \frac{\partial}{\partial z}(-3z) = 0$$

$$\Rightarrow \alpha + 2 - 3 = 0$$

$$\Rightarrow \alpha = 1$$

14. A rod of length L with uniform charge density λ per unit length is in the xy -plane and rotating about z -axis passing through one of its edge with an angular velocity $\vec{\omega}$ as shown in the figure below. $(\hat{r}, \hat{\phi}, \hat{z})$ refer to the unit vectors at Q , $\vec{A}$ is the vector potential at a distance d from the origin O along z -axis for $d \gg L$ and $\vec{J}$ is the current density due to the motion of the rod. [GATE 2008]



- (a) $\vec{J}$ along $\hat{r}$; $\vec{A}$ along $\hat{z}$; $|\vec{A}| \propto \frac{1}{d}$
- (b) $\vec{J}$ along $\hat{\phi}$; $\vec{A}$ along $\hat{\phi}$; $|\vec{A}| \propto \frac{1}{d^2}$
- (c) $\vec{J}$ along $\hat{r}$; $\vec{A}$ along $\hat{z}$; $|\vec{A}| \propto \frac{1}{d^2}$
- (d) $\vec{J}$ along $\hat{\phi}$; $\vec{A}$ along $\hat{\phi}$; $|\vec{A}| \propto \frac{1}{d}$

Soln. Since rod is rotated in the xy -plane the direction of $\vec{J}$ and $\vec{A}$ will be along $\hat{\phi}$.
Since, $d \gg L$, so we can consider rod as magnetic dipole, the vector potential due to magnetic dipole as

$$\vec{A} = \frac{\mu_0}{4\pi} \frac{\vec{M} \times \vec{d}}{d^3}$$

$$\vec{A} \propto \frac{1}{d^2}$$

Correct option is (b)

15. The magnetic field (in $A \text{ m}^{-1}$) inside a long solid cylindrical conductor of radius $a = 0.1 \text{ m}$ is,

$$H = \frac{10^4}{r} \left[\frac{1}{\alpha^2} \sin(\alpha r) - \frac{r}{\alpha} \cos(\alpha r) \right] \hat{\phi}, \text{ where } \alpha = \frac{\pi}{2a}. \text{ What is the total current (in A) in the conductor?}$$

- (a) $\frac{\pi}{2a}$ (b) $\frac{800}{\pi}$ (c) $\frac{400}{\pi}$ (d) $\frac{300}{\pi}$ [GATE 2009]

Soln. $\vec{J} = (\vec{\nabla} \times \vec{H})$

$$= \frac{1}{r} \frac{\partial}{\partial r} \left[10^4 \left(\frac{1}{\alpha^2} \sin(\alpha r) - \frac{r}{\alpha} \cos(\alpha r) \right) \right] \hat{k}$$

$$= \frac{10^4}{r} \left[\frac{1}{\alpha} \cos \alpha r - \frac{1}{\alpha} \cos(\alpha r) + r \sin(\alpha r) \right] \hat{k}$$

$$= 10^4 \sin(\alpha r) \hat{k}$$

$$I = \int \vec{J} \cdot d\vec{s}$$

$$= 10^4 \int_0^a \sin \alpha r \cdot 2\pi r dr = 10^4 2\pi \int_0^a \sin(\alpha r) r dr$$

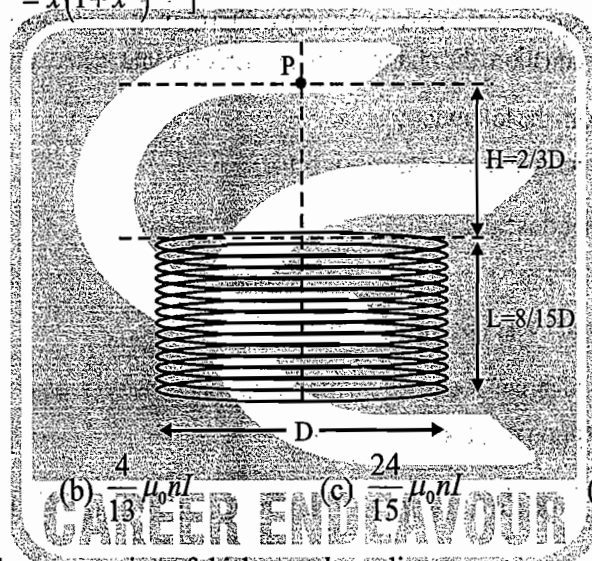
$$= 2\pi \times 10^4 \left[-\frac{(\cos(\alpha r)) \cdot r}{\alpha} + \frac{\sin(\alpha r)}{\alpha^2} \right]_0^a = 2\pi \times 10^4 \left[-\frac{\cos \alpha a}{\alpha} \cdot a + \frac{\sin \alpha a}{\alpha^2} \right]$$

$$= 2\pi \times 10^4 \left[-\frac{\cos \frac{\pi}{2}}{\alpha} \cdot a + \frac{\sin \frac{\pi}{2}}{\left(\frac{\pi^2}{4a^2} \right)} \right] = \frac{8a^2}{\pi} \times 10^4 = \frac{800}{\pi}$$

Correct option is (b)

16. A short solenoid with n turns per unit length has diameter D and length $L = 8D/15$, as shown in the figure, and it carries a constant current I . The magnetic field B at a point P on the axis of the solenoid at a distance $H = 2D/3$ from its near end (see figure) is [TIFR 2014]

[use $\int dx(1+x^2)^{-3/2} = x(1+x^2)^{-1/2}$]



(a) $\frac{4}{65} \mu_0 n I$

(b) $\frac{4}{13} \mu_0 n I$

(c) $\frac{24}{15} \mu_0 n I$

(d) $\frac{112}{65} \mu_0 n I$

Soln. Let consider an elementary ring of thickness dx at distance x .

$$dB = \frac{\mu_0 I n dx a^2}{2(a^2 + x^2)^{3/2}}$$

$$B = \frac{\mu_0 I n}{2} \int_H^{H+L} \frac{a^2 dx}{(a^2 + x^2)^{3/2}} = \frac{\mu_0 I n}{2} \int_{2/3D}^{6/5D} \frac{(D/2)^2 dx}{((D/2)^2 + x^2)^{3/2}}$$

$$= \frac{\mu_0 I n}{2} \int_{4/3}^{12/5} \frac{dz}{(1+z^2)^{3/2}} = \frac{\mu_0 I n}{2} \left[\frac{z}{\sqrt{1+z^2}} \right]_{4/3}^{12/5}$$

$$= \frac{\mu_0 I n}{2} \left[\frac{12/5}{\sqrt{1+\frac{144}{25}}} - \frac{4/3}{\sqrt{1+\frac{16}{9}}} \right] = \frac{\mu_0 I n}{2} \left[\frac{12/5}{13/5} - \frac{4/3}{5/3} \right]$$

$$= \frac{\mu_0 I n}{2} \left(\frac{12}{13} - \frac{4}{5} \right) = \frac{\mu_0 I n}{2} \left(\frac{60-52}{65} \right) = \frac{4}{65} \mu_0 I n$$



PRACTICE SET

- A dipole of magnetic moment $\vec{m}$ is placed in an inhomogeneous magnetic field $\vec{B}$. The force experienced by the dipole is :
 (a) $\vec{F} = \vec{m} \times (\vec{\nabla} \times \vec{B})$ (b) $\vec{F} = (\vec{m} \cdot \vec{\nabla}) \vec{B}$ (c) $\vec{F} = (\vec{\nabla} \cdot \vec{m}) \vec{B}$ (d) $\vec{F} = \vec{\nabla} \times (\vec{m} \times \vec{B})$
- A very long solenoid with n turns per unit length carries a current I . The magnetic field at a point, which is on its axis and its end face, is :
 (a) $\mu_0 n I$ (b) $(2/3) \mu_0 n I$ (c) $(1/3) \mu_0 n I$ (d) $(1/2) \mu_0 n I$
- If in interior of a unit sphere we have $\vec{\nabla} \cdot \vec{J} = a$ positive constant, where $\vec{J}$ is the current density then we may conclude that;
 (a) According to Gauss' theorem, the charge contained in the unit sphere is constant in time.
 (b) According to Gauss' theorem, charge is flowing into the unit sphere.
 (c) According to the continuity equation, the charge density within the unit sphere must necessarily be uniform.
 (d) According to the continuity equation, the charge density inside the unit sphere diminishes with time.
- Choose the correct statement from the following:
 (a) The magnetic field $\vec{B}$ is a vector and not a pseudovector.
 (b) $\vec{E} \cdot \vec{B}$ is a scalar and not a pseudoscalar.
 (c) $\vec{E} \times \vec{B}$ is a pseudovector.
 (d) The magnetic vector potential $\vec{A}$ is a pseudovector.
- A wire of circular cross section of radius R carries a volume current density $\vec{J} = k r^2 \hat{z}$ along the length of the wire. The magnetic field at a position $r < R$ is
 (a) $\frac{\mu_0}{2} k r^3 \hat{z}$ (b) $\frac{\mu_0}{4} k r^3 \hat{\phi}$ (c) $\frac{\mu_0}{4\pi} k r^3 \hat{r}$ (d) $\frac{\mu_0}{4\pi} k r^2 \hat{\phi}$
- Consider an infinite plane at $z = 0$ with uniform surface current density $\vec{K} = K \hat{i}$. The direction of the magnetic field $\vec{B}$ and the vector potential $\vec{A}$ of this current distribution at a distance d along the positive z -axis are :
 (a) $\vec{B}$ along $\hat{j}$ and $\vec{A}$ along $\hat{k}$ (b) $\vec{B}$ along $\hat{k}$ and $\vec{A}$ along $\hat{j}$
 (c) $\vec{B}$ along $-\hat{j}$ and $\vec{A}$ along $\hat{i}$ (d) $\vec{B}$ along $\hat{i}$ and $\vec{A}$ along $-\hat{j}$
- The magnitude of vector potential at a point r ($r < R$) in an infinite solenoid with N -turns per unit length, radius R and carrying a current I , is given by
 (a) $A = \mu_0 N I r^2$ (b) $A = \frac{\mu_0 N I r^2}{2}$ (c) $A = \mu_0 N I r$ (d) $A = \frac{\mu_0 N I r}{2}$
- Magnetic induction $|B|$ due to current I in a coil of radius R placed in y - z plane at a distance X along the axis of the coil is given by $\frac{\mu_0 I}{2} \frac{R^2}{(R^2 + X^2)^{3/2}}$. Another similar coil of radius R , carrying equal current I is placed in the x - z plane at a distance $2R$ from the first coil along the same axis. The value of $|B|$ at a point midway

Magnetostatics

(a) $\frac{\mu_0 I}{2\sqrt{2} R}$

(b) $\frac{\mu_0 I R}{2\sqrt{2}}$

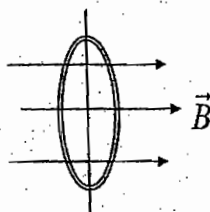
(c) 0

(d) $\frac{\mu_0 I}{4\sqrt{2} R}$

The expression $\vec{A} = \frac{B}{2}(-\hat{y} + \hat{j}x)$ for the vector potential represents

- (a) a uniform magnetic field in the z-direction (b) a non-uniform magnetic field in the z-direction
(c) a uniform magnetic field in xy-plane (d) a non-uniform magnetic field in xy-plane

A loop of wire of radius a and carrying a current I is suspended vertically in a uniform horizontal magnetic field B as shown. Calculate the work done in rotating the loop around the vertical axis through an angle π



(a) $\frac{\pi^2 I^2}{2a}$

(b) $2\pi a^2 I$

(c) $\frac{\pi a^2 I}{2}$

(d) 0

ANSWER KEY

Questions	1	2	3	4	5
Option	(b)	(d)	(d)	(b)	(b)
Questions	6	7	8	9	10
Option	(c)	(d)	(a)	(a)	(b)

CAREER ENDEAVOUR

Chapter 10

Magnetization & Magnetic Materials

Introduction :

So far we have read the magnetic field $\vec{B}$ produced by the current in vacuum. In this chapter we shall discuss the effect of magnetic field inside the medium.

Magnetization

Any current loop has a magnetic field and thus has a magnetic dipole moment, including the atomic-level current loops described in some models of the atom. Thus, the magnetic moments in a magnetized substance may be described as arising from these atomic-level current loops. For the Bohr model of the atom, these current loops are associated with the movement of electrons around the nucleus in circular orbits. In addition, a magnetic moment is intrinsic to electrons, protons, neutrons, and other particles; it arises from a property called spin.

Ordinarily, the current loops inside an object cancel each other out because of the random orientation of the atoms. But when a magnetic field is applied, a net alignment of these magnetic dipoles occurs, and the medium becomes magnetically polarized, or magnetized.

Let's define,

$\vec{M} \equiv$ magnetic dipole moment per unit volume

$$\vec{M} = \lim_{\Delta v \rightarrow 0} \frac{1}{\Delta v} \sum_{i=1}^N \vec{m}_i$$

$\vec{M}$ is called the magnetization; it plays a role analogous to the polarization $\vec{P}$ in electrostatics. Right now the equation is how the magnetization got there it could be paramagnetism, diamagnetism, or even ferromagnetism. We shall take $\vec{M}$ as given, and calculate the field this magnetization itself produces.

Volume and surface current

In electric field we have shown the occurrence of bound charges (volume and surface bound charges). While calculating the field of a magnetized object the bound currents come. There are bound surface current and volume current. We won't derive the expression –

Field of a magnetized object is

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{1}{r} [\nabla \times \vec{M}(\vec{r}')] d\tau' + \frac{\mu_0}{4\pi} \oint \frac{1}{r} [\vec{M}(\vec{r}') \times d\vec{a}']$$

The first term looks just like the potential of a volume current,

$$\vec{J}_b = \nabla \times \vec{M} \quad \dots (3)$$

while the second looks like the potential of a surface current,

$$\vec{K} = \vec{M} \times \hat{n} \quad \dots (4)$$



$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int_V \frac{\vec{J}_b(\vec{r}')}{r} d\tau' + \frac{\mu_0}{4\pi} \int_S \frac{\vec{K}_b(\vec{r}')}{r} da' \quad \dots (5)$$

Magnetizing field $\vec{H}$

As we have seen, there are both free and bound current. So the total current $\vec{J}$ could be written as

$$\vec{J} = \vec{J}_b + \vec{J}_f \quad \dots (6)$$

Where $\vec{J}_b$ is bound and $\vec{J}_f$ is the free current.

We know the Ampere's Law tells us

$$\oint \vec{B} \cdot d\vec{\ell} = \mu_0 I_{enc.}$$

And,
$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$$

So from there we replace the value of $\vec{J}$ and use

$$\vec{J}_b = \vec{\nabla} \times \vec{M}$$

$$\frac{1}{\mu_0} (\vec{\nabla} \times \vec{B}) = \vec{J} = \vec{J}_f + \vec{J}_b = \vec{J}_f + (\vec{\nabla} \times \vec{M})$$

Or,
$$\vec{\nabla} \times \left(\frac{1}{\mu_0} \vec{B} - \vec{M} \right) = \vec{J}_f \quad \dots (7)$$

Now, define the quantity inside the curl as $\vec{H}$

$$\vec{H} \equiv \frac{1}{\mu_0} \vec{B} - \vec{M} \quad \dots (8)$$

So now the equation becomes

$$\vec{\nabla} \times \vec{H} = \vec{J}_f \quad \dots (9)$$

Or in integral terms

$$\oint \vec{H} \cdot d\vec{\ell} = I_{fenc.} \quad \dots (10)$$

Where $I_{fenc.}$ is the total free current passing through the Amperian loop. These are the forms of Ampere's law inside matter. $\vec{H}$ plays a role in magnetostatics analogous to $\vec{D}$ in electrostatics:

Linear and non-linear medium

For linear medium $\vec{M}$ is proportional to $\vec{H}$

$$\vec{M} = \chi_m \vec{H} \quad \dots (11)$$

The constant of proportionality χ_m is called the magnetic susceptibility; it is a dimensionless quantity that varies from one substance to another—positive for paramagnets and negative for diamagnets. Typical values are around 10^{-5} .

From
$$\vec{H} \equiv \frac{1}{\mu_0} \vec{B} - \vec{M}$$
, we get

$$\vec{B} = \mu_0 (\vec{H} + \vec{M}) = \mu_0 (1 + \chi_m) \vec{H} \quad \dots (12)$$

Or defining,
$$\mu \equiv \mu_0 (1 + \chi_m) \quad \dots (13)$$

μ is called the permeability of the material. In a vacuum, where there is no matter to magnetize, the susceptibility χ_m vanishes, and the permeability is μ_0 . That's why μ_0 is called the permeability of free space.

Magnetic moment of atoms

Electrons not only spin; they also revolve around the nucleus—for simplicity, let's assume the orbit is a circle of radius R (Fig. 6.1). Although technically this orbital motion does not constitute a steady current, in practice the period $T = 2\pi R / v$ is so short that unless you blink awfully fast, it's going to look like a steady current:

$$I = \frac{e}{T} = \frac{ev}{2\pi R}$$

The orbital dipole moment is: $\vec{m} = IA = I\pi R^2$

$$\vec{m} = -\frac{1}{2}evR\hat{z}$$

The minus sign accounts for the negative charge of the electron. Like any other magnetic dipole, this one is subject to a torque ($\vec{m} \times \vec{B}$) when the atom is placed in a magnetic field.

Diamagnet - paramagnet - ferromagnet

Unlike electric polarization, which is almost always in the same direction as $\vec{E}$, some materials acquire a magnetization parallel to $\vec{B}$ (paramagnets) and some opposite to $\vec{B}$ (diamagnets). A few substances (called ferromagnets, in deference to the most common example, iron) retain their magnetization even after the external field has been removed.

For these (ferromagnets) the magnetization is not determined by the present field but by the whole magnetic "history" of the object. Permanent magnets made of iron are the most familiar examples of magnetism, though from a theoretical point of view they are the most complicated. Let's begin with paramagnetism and diamagnetism.

For details study about Diamagnetic, Paramagnetic and Ferromagnetic material please read any solid state book.

Magnetostatic Boundary Condition

Like the electrostatic case we have magnetostatic boundary conditions too. But the conditions are not same. For magnetostatics the boundary conditions are

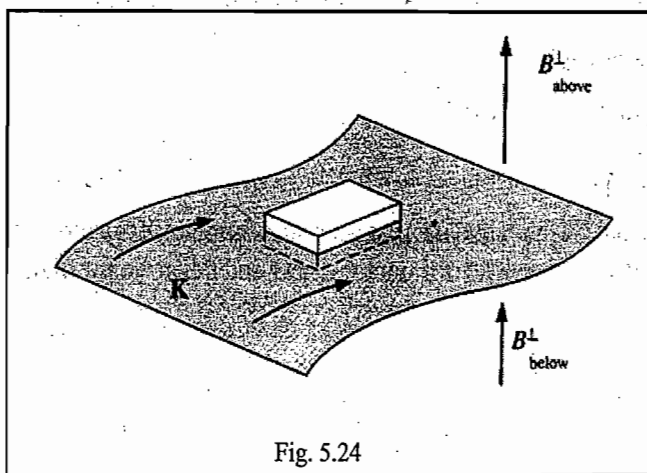


Fig. 5.24

$$\vec{B}_{\text{above}}^{\perp} = \vec{B}_{\text{below}}^{\perp}$$

... (15)

Combined form of equation (15) and (16) is given as

$$\vec{B}_{\text{above}} - \vec{B}_{\text{below}} = \mu_0 (\mathbf{K} \times \hat{n}) \quad \dots (17)$$

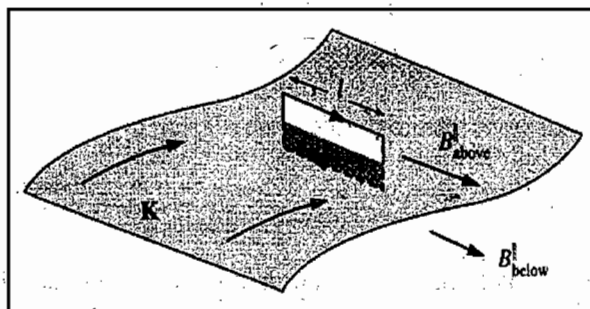
Where $\hat{n}$ is a unit vector perpendicular to the surface, pointing "upward".

Like the scalar potential of electrostatics the magnetic vector potential is also continuous

$$A_{\text{above}} = A_{\text{below}} \quad \dots (18)$$

But the derivative of vector potential has a discontinuity

$$\frac{\partial A_{\text{above}}}{\partial n} - \frac{\partial A_{\text{below}}}{\partial n} = -\mu_0 K \quad \dots (19)$$



SOLVED PROBLEMS

1. An infinitely long circular cylinder carries a uniform magnetization $\mathbf{M}$ parallel to its axis. Find the magnetic field (due to $\mathbf{M}$) inside and outside the cylinder.

Soln. $\vec{J}_b = \vec{\nabla} \times \vec{M} = 0$; $\vec{K}_b = \vec{M} \times \hat{n} = M\hat{\phi}$



The field is that of a surface current $\vec{K}_b = M\hat{\phi}$ but that's just a solenoid, so the field **outside is zero**, and inside $\mathbf{B} = \mu_0 \mathbf{K}_b = \mu_0 \mathbf{M}$. Moreover, it points upward (in the drawing).

2. A long copper rod of radius R carries a uniformly distributed free current I . Find $\mathbf{H}$ inside and outside the rod

Soln. Take an Amperian loop of radius $s < R$. Apply

$$\oint \vec{H} \cdot d\vec{\ell} = I_{\text{fenc.}}$$

We get (as enclosed current is not I , but fraction of I enclosed by the loop)

$$H(2\pi s) = I_{\text{fenc}} = I \frac{\pi s^2}{\pi R^2}$$

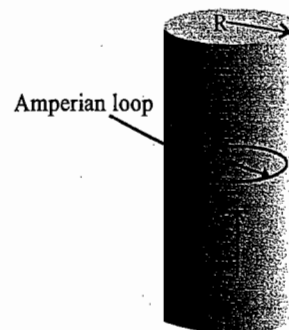
$$H = \frac{I}{2\pi R^2} s \hat{\phi} \quad (s \leq R)$$

Outside of the wire ($s \geq R$) the situation is different (Enclosed current = I)

$$H = \frac{I}{2\pi s} \hat{\phi} \quad (s \geq R)$$

Outside, (as always, in empty space) $\mathbf{M} = 0$, so

$$\mathbf{B} = \mu_0 \mathbf{H} = \frac{\mu_0 I}{2\pi s} \hat{\phi} \quad (s \geq R)$$





3. Consider a plane interface of a magnetic medium of permeability μ and vacuum. If the magnetic field $\vec{B}$ in the medium in the vicinity of a point on the interface makes an angle θ with the normal to the interface, find the magnetic field $\vec{B}_0$ in vacuum in the vicinity of the above point.

Soln. Continuity of the normal component of $\vec{B}$ indicates that $B_{0n} = B \cos \theta$.

From the continuity of the tangential component of $\vec{H}$ we get

$$H_{0t} = H_t \text{ or } \frac{B_{0t}}{\mu_0} = \frac{B_t}{\mu} \text{ or } B_{0t} = \frac{\mu_0 B_t}{\mu} = \mu_0 B \sin \theta / \mu$$

Therefore, the magnitude of $\vec{B}$ -field in vacuum would be

$$B_0 = \sqrt{B_{0n}^2 + B_{0t}^2} = B \sqrt{\cos^2 \theta + \mu_0^2 \sin^2 \theta / \mu^2}$$

The angle θ_0 made by $\vec{B}_0$ with the normal to the interface is given by

$$\tan \theta_0 = \frac{B_{0t}}{B_{0n}} = \frac{\mu_0}{\mu} \tan \theta$$

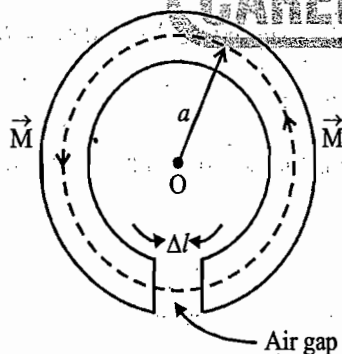
4. Consider a thin permanently magnetized toroid of mean radius a having a small air gap $\Delta \ell$ between its pole faces. If it has a magnetization $\vec{M}$ of constant magnitude in circumferential direction find $\vec{B}$ and $\vec{H}$ both in the air gap and within the material.

Soln. Applying Ampere's circuital law to the dotted path in figure.

$$\oint \vec{H} \cdot d\vec{\ell} = 0 \text{ or } H_i (2\pi a - \Delta \ell) + H_0 \cdot \Delta \ell = 0$$

Now we note that the magnetic flux (BA) or flux density (B) may be assumed to be the same throughout. Magnetization inside the air gap is zero. Therefore, we can write,

$$H_0 = \frac{B}{\mu_0} \text{ and } H_i = \frac{B}{\mu_0} - M$$



Therefore,

$$\left(\frac{B}{\mu_0} - M \right) (2\pi a - \Delta \ell) + \frac{B}{\mu_0} \cdot \Delta \ell = 0 \text{ or } B = \frac{\mu_0 M (2\pi a - \Delta \ell)}{2\pi a}$$

both in the material and in the air gap. Now the magnetic intensity in air gap is $H_0 = \frac{B}{\mu_0}$ and in the material,

$$H_i = \frac{B}{\mu_0} - M = -\frac{M \cdot \Delta \ell}{2\pi a}$$



PRACTICE SET

1. The intrinsic magnetic moment of the atoms of a material is not zero. The material
 - (a) must be paramagnetic
 - (b) must be diamagnetic
 - (c) must be ferromagnetic
 - (d) may be paramagnetic or ferromagnetic
2. The relative permeability of a material is 0.98. The material must be
 - (a) paramagnetic
 - (b) ferromagnetic
 - (c) diamagnetic
 - (d) none of these
3. Susceptibility χ of a paramagnetic gas varies with temperature as
 - (a) $\chi \propto T$
 - (b) $\chi \propto T^{-1}$
 - (c) $\chi \propto T^{-2}$
 - (d) $\chi \propto T^0$
4. Which of the following statements is not applicable to ferromagnetic materials?
 - (a) susceptibility $\chi_m \gg 1$
 - (b) it shows the phenomenon of hysteresis
 - (c) its permeability has a fixed value
 - (d) it can show spontaneous magnetization below certain temperature
5. When a ferromagnetic material is taken through a hysteresis loop its susceptibility (χ) defined through $\vec{M} = \chi \vec{H}$ may not take up a value which is
 - (a) zero
 - (b) negative
 - (c) infinity
 - (d) constant x

ANSWER KEY

Questions	1	2	3	4	5
Option	(d)	(c)	(b)	(c)	(d)

CAREER ENDEAVOUR

Chapter 11

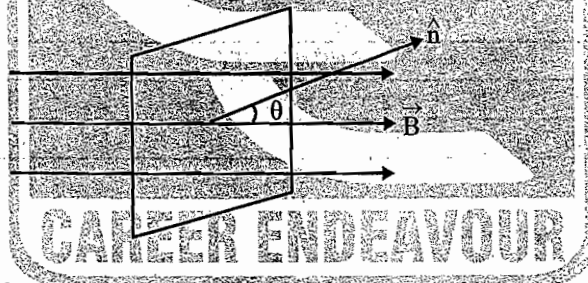
Faraday's law & Electromagnetic Induction

Introduction :

In 1931 Michael Faraday performed experiments to check whether current is produced in a closed wire placed near magnetic fields. He found no current in that case but he realized that a time-varying current in the loop is obtained while the magnet is being moved towards or away from it. The law he formulated is known as Faraday's law of electromagnetic induction. It is most important law of electromagnetic induction. It is most important law of electromagnetism. Without it there would be no electricity from rotating generators, no telephone, no radio and television.

Magnetic flux:

Consider a uniform magnetic field passing through a surface s , as shown in figure below.



Let the area vector be $\vec{A} = A\hat{n}$, where A is the area of the surface and $\hat{n}$ its unit normal. The magnetic flux through the surface is given by

$$\phi_B = \vec{B} \cdot \vec{A} = BA \cos \theta \quad \dots (1.a)$$

Where, θ is the angle between A and B .

For non-uniform magnetic field, ϕ is given by

$$\phi = \iint_S \vec{B} \cdot d\vec{A} \quad \dots (1.b)$$

Faraday's Laws of Electromagnetic Induction:

- (i) Induced emf in a circuit is proportional to the rate of change of magnetic flux linked with the circuit.
- (ii) The direction of induced emf is such that it tries to oppose the cause of its generation, i.e., the variation of magnetic flux producing it.

If Φ be the magnetic flux linked with a circuit at any time 't' then the laws of electromagnetic induction can be expressed mathematically as

$$\varepsilon = - \frac{d\Phi}{dt} \quad \dots (1.c)$$

where ε is the induced emf



Self-Induction:

When a current flows in a circuit it produces a magnetic field in the surrounding space and hence there will be a magnetic flux linked with the circuit. If the current through the circuit varies with time the flux linked with the circuit also varies with time and gives rise to an induced emf in the circuit. This phenomenon is known as self-inductance. According to Lenz's law the induced emf opposes the change in current in the circuit. For this reason it is called a back emf. Self-inductance plays somewhat the same role in electrical circuits that mass plays in mechanical system. If the circuit is in a non-ferromagnetic medium the flux Φ linked with it will be proportional to the current I in the circuit. Thus,

$$\Phi = LI \quad \dots (2)$$

where L is a constant, which depends on the geometry of the circuit, its dimensions and the permeability of the surrounding medium. The quantity L is called the coefficient of self-inductance or simply the self-inductance of the circuit. Numerically self-inductance may be defined as the total flux linked with the circuit when the unit current flows through it.

The self induced emf in the circuit is given by

$$\varepsilon = -\frac{d\Phi}{dt} = -\frac{d}{dt}(LI)$$

If the medium is not ferromagnetic and the circuit is rigid then L is a constant and we can write.

$$\varepsilon = -L \frac{dI}{dt} \quad \dots (3)$$

Thus, self-inductance may also be defined as numerically equal to the emf induced in the circuit for a unit rate of change of current in it.

In case of ferromagnetic material Φ is not linearly related to I . Then for a rigid stationary circuit where flux change is caused by change in current only, we can write

$$\varepsilon = -\frac{d\Phi}{dt} = -\frac{d\Phi}{dI} \frac{dI}{dt} = -L \frac{dI}{dt} \quad \dots (4)$$

where $L = \frac{d\Phi}{dI}$ is called incremental self-inductance.

Calculation of self-inductance in simple cases:

(1) A solenoid :

Suppose we consider an air-cored long solenoid of length l containing N number of turns. If a current I flows through it the axial magnetic field at any inside point is given by

$$B = \mu_0 n I = \frac{\mu_0 N I}{l}$$

If A be the area of cross-section of the solenoid and the magnetic field B is assumed to have the same value over the cross-section then total magnetic flux linking the N -turns of the solenoid would be

$$\Phi = N \cdot B \cdot A = \frac{\mu_0 N^2 A}{l} I$$

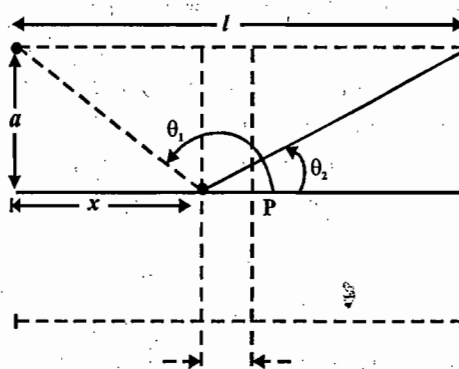
Now the self-inductance L is defined as the flux linkage per unit current. Therefore, the self-inductance of the solenoid is

$$L = \frac{\Phi}{I} = \frac{\mu_0 N^2 A}{l} \quad \dots (5)$$



$$L = \frac{\mu N^2 A}{\ell}$$

(for material having constant permeability μ)



If the solenoid is not very long then the axial magnetic field at any axial point P is given by

$$B = \frac{\mu_0 NI}{2} (\cos \theta_2 - \cos \theta_1) = \frac{\mu_0 NI}{2} \left[\frac{l-x}{\sqrt{(l-x)^2 + a^2}} + \frac{x}{\sqrt{x^2 + a^2}} \right]$$

In a length dx about P there are $(N/l)dx$ number of turns and hence the flux linking these turns is

$$d\phi = \left(\frac{N}{l} dx \right) BA$$

In a length dx about P there are $\left(\frac{N}{l} \right) dx$ number of turns and hence the flux linking these turns is

$$d\Phi = \left(\frac{N}{l} dx \right) BA$$

where, we assume that B is the uniform over the entire cross-section.

Therefore, total magnetic flux linking the solenoid is

$$\Phi = \int d\Phi = \int_0^l \frac{BAN}{l} dx = \frac{\mu_0 N^2 AI}{2\ell^2} \int_0^l \left[\frac{\ell-x}{\sqrt{(\ell-x)^2 + a^2}} + \frac{x}{\sqrt{x^2 + a^2}} \right] dx$$

$$\text{Or, } \Phi = \frac{\mu_0 N^2 AI}{2\ell^2} \left[-\sqrt{(\ell-x)^2 + a^2} + \sqrt{x^2 + a^2} \right]_0^l$$

$$= \frac{\mu_0 N^2 A}{\ell^2} \left[\sqrt{\ell^2 + a^2} - a \right] \cdot I$$

$$\text{Therefore, } L = \frac{\mu_0 N^2 A}{\ell^2} \left[\sqrt{\ell^2 + a^2} - a \right]$$

$$= \frac{\mu_0 N^2 A}{\ell} \left[\sqrt{1 + (a/\ell)^2} - a/\ell \right] \quad \dots (6)$$

(2) A toroidal coil:

The magnetic field inside a toroidal coil having N number of turns, mean length ℓ , cross-sectional area A and carrying current I can be written as

$$B = \frac{\mu_0 NI}{\ell}$$



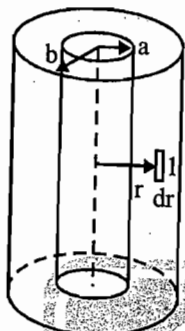
Neglecting any variation of B over the cross-sectional area we can write for the flux linking the N turns of the toroid as

$$\Phi = N \cdot B \cdot A = \frac{\mu_0 N^2 A}{\ell} I$$

Therefore, the self-inductance is

$$L = \frac{\Phi}{I} = \frac{\mu_0 N^2 A}{\ell} \quad \dots (7)$$

(3) Long coaxial cable:



Consider a long coaxial cable consisting of two concentric cylinders. The inner one has radius a and carries a current I while the outer one has radius b and carries a return current I . Applying Ampere's circuital law it is easy to show that the magnetic field outside the cable is zero and at an internal point at a distance r from the axis ($a < r < b$) the field is given by

$$B = \frac{\mu_0 I}{2\pi r}$$

The lines of $\vec{B}$ are circles about the axis of the cable. If we consider an elemental area of radial width dr and axial length unity then flux through it would be

$$d\Phi = B \cdot dr \times 1 = \frac{\mu_0 I}{2\pi r} dr$$

Therefore, total flux per unit axial length is

$$\Phi = \int d\Phi = \frac{\mu_0 I}{2\pi} \int_a^b \frac{dr}{r} = \frac{\mu_0 I}{2\pi} \ln \frac{b}{a}$$

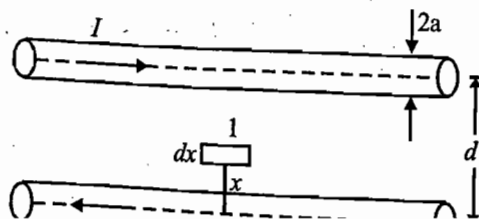
So, the self-inductance per unit length is

$$L = \frac{\Phi}{I} = \frac{\mu_0}{2\pi} \ln \frac{b}{a} \quad \dots (8)$$

In the above discussion we have neglected the flux within the materials of the two cylinders. This is justified when $b \gg a$.

(4) Two-wire transmission line:

Let us consider two long parallel wires separated by a distance d in air. The wires carrying the same current I in opposite directions. We shall assume that the radius a of each wire is much less than d , so that the flux inside the material of the wires may





be neglected. The flux lines will be mainly concentrated in the region between the wires. Total magnetic field at any point at a distance x from one wire is

$$B = \frac{\mu_0 I}{2\pi} \left[\frac{1}{x} + \frac{1}{d-x} \right]$$

The flux through an elemental area of width dx and length unity is

$$d\Phi = B \cdot dx \times 1 = \frac{\mu_0 I}{2\pi} \left[\frac{1}{x} + \frac{1}{d-x} \right] dx$$

Hence, the total flux through the entire area between the two wires of unit length is

$$\begin{aligned} \Phi &= \int d\Phi = \frac{\mu_0 I}{2\pi} \int_a^{d-a} \left[\frac{1}{x} + \frac{1}{d-x} \right] dx \\ &= \frac{\mu_0 I}{2\pi} \left[\ln \frac{d-a}{a} - \ln \frac{a}{d-a} \right] = \frac{\mu_0 I}{\pi} \ln \frac{d-a}{a} \end{aligned}$$

Therefore, self-inductance per unit length is

$$L = \frac{\Phi}{I} = \frac{\mu_0}{\pi} \ln \frac{d-a}{a} \quad \dots (10)$$

Mutual Induction:

Suppose two coils are placed near each other, as shown in the figure below.

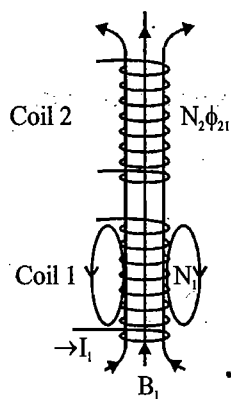


Figure-1

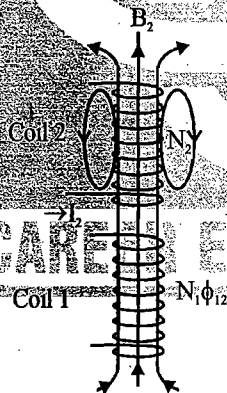


Figure-2

The first coil has N_1 turn and carries a current I_1 which gives rise to a magnetic field $\vec{B}_1$. Since the two coils are close to each other, some of the magnetic field lines through coil 1 will also pass through coil 2. Let ϕ_{21} denote the magnetic flux through one turn of coil 2 due to I_1 . Now, by varying I_1 with time, there will be an induced emf associated with the changing magnetic flux in second coil.

$$\varepsilon_{21} = -N_2 \frac{d\phi_{21}}{dt} = -\frac{d}{dt} \iint \vec{B}_1 \cdot d\vec{A}_2$$

The time rate of change of magnetic flux ϕ_{21} in coil 2 is proportional to the time rate of change of the current in coil 1.

$$N_2 \frac{d\phi_{21}}{dt} = M_{21} \frac{dI_1}{dt}$$



$$M_{21} = \frac{N_2 \phi_{21}}{I_1}$$

In similar manner suppose instead there is a current I_2 in the second coil and it is varying with time. Then the induced emf in coil 1 becomes

$$\varepsilon_{12} = -N_1 \frac{d\phi_{12}}{dt} = -\frac{d}{dt} \iint \vec{B}_2 \cdot d\vec{A}_1$$

and a current is induced in coil 1.

This changing flux in coil 1 is proportional to the changing current in coil 2.

$$N_1 \frac{d\phi_{12}}{dt} = M_{12} \frac{dI_2}{dt}$$

$$\Rightarrow M_{12} = \frac{N_1 \phi_{12}}{I_2}$$

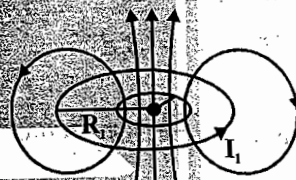
However, using the reciprocity theorem which combine Ampere's law and Biot-Savart law, one may show that the constants are equal

$$M_{12} = M_{21} = M$$

Calculation of Mutual inductances in simple cases:

(1) Consider two single-turn co-planar, concentric coils of radii R_1 and R_2 with $R_1 \gg R_2$ as shown in figure. The outer ring carries current I_1 . So, the magnetic field at the centre is

$$B_1 = \frac{\mu_0 I_1}{2R_1}$$



Since, $R_1 \gg R_2$, we appropriate the magnetic field through the entire inner coil by B_1 . Hence, the flux through the second coil is

$$\phi_{21} = B_1 A_2 = \frac{\mu_0 I_1}{2R_1} \pi R_2^2 = \frac{\mu_0 \pi I_1 R_2^2}{2R_1}$$

Thus, the mutual inductance is given by

$$M = \frac{\phi_{21}}{I_1} = \frac{\mu_0 \pi R_2^2}{2R_1}$$

The result shows that M depends only on the geometrical factors, R_1 and R_2 , and is independent of the current I_1 in the coil.

Energy in the magnetic field

The work done on a unit charge, against the back emf, in one trip around the circuit is $-\oint \vec{E} \cdot d\vec{l}$ (the minus sign records the fact that this is the work done by you against the emf, not the work done by the emf). The amount of charge per unit time passing down the wire is I . So the total work done per unit time is

$$\frac{dW}{dt} = -\varepsilon I = LI \frac{dI}{dt}$$

Hence, integrating we get

$$W = \frac{1}{2} LI^2$$

... (11)



We can prove that

$$W = \frac{1}{2\mu_0} \int_{\text{all space}} B^2 d\tau \quad \dots (12)$$

(Here “ $d\tau$ ” is the volume element “ dv ”)

So, summarizing energy stored in electric and magnetic field are

$$\left. \begin{aligned} W_{\text{elec}} &= \frac{1}{2} \int (V\rho) d\tau = \frac{\epsilon_0}{2} \int E^2 d\tau \\ W_{\text{mag}} &= \frac{1}{2} \int (\vec{A} \cdot \vec{J}) d\tau = \frac{1}{2\mu_0} \int B^2 d\tau \end{aligned} \right\} \quad \dots (13)$$

SOLVED PROBLEMS

1. A thin conducting wire is bent into a circular loop of radius r and placed in a time dependent magnetic field of magnetic induction. [GATE 2001]

$$\vec{B}(t) = B_0 e^{-\alpha t} \hat{e}_z, \quad (B_0 > 0 \text{ and } \alpha > 0),$$

such that, the plane of the loop is perpendicular to $\vec{B}(t)$. Then the induced emf in the loop is :

- (a) $\pi^2 \alpha B_0 e^{-\alpha t}$ (b) $\pi^2 B_0 e^{-\alpha t}$ (c) $-\pi^2 \alpha B_0 e^{-\alpha t}$ (d) $-\pi^2 B_0 e^{-\alpha t}$

Soln. The flux through the circular disc is

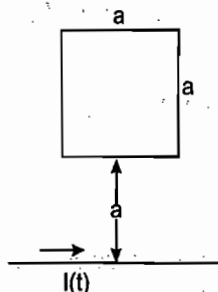
$$\phi = \pi r^2 B(t) = \pi r^2 B_0 e^{-\alpha t}$$

$$\text{Therefore, the induce emf} = \frac{d\phi}{dt} = -\pi r^2 \alpha B_0 e^{-\alpha t}$$

Correct option is (a)

2. An infinitely long wire carrying a current $I(t) = I_0 \cos(\omega t)$ is placed at a distance a from a square loop of side a as shown in the figure. If the resistance of the loop is R , then the amplitude of the induced current in the loop is : [GATE 2005]

- (a) $\frac{\mu_0 a I_0 \omega}{2\pi R} \ln 2$ (b) $\frac{\mu_0 a I_0 \omega}{\pi R} \ln 2$
(c) $\frac{2\mu_0 a I_0 \omega}{\pi R} \ln 2$ (d) $\frac{\mu_0 a I_0 \omega}{2\pi R}$

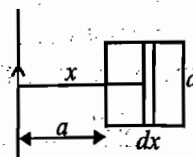


Soln. $B = \frac{\mu_0 I}{2\pi x}$

$$d\phi = B a dx = \frac{\mu_0 I}{2\pi x} a dx$$

$$= \frac{\mu_0 I a}{2\pi} \int_a^{2a} \frac{dx}{x} = \frac{\mu_0 I a}{2\pi} \ln \left(\frac{2a}{a} \right) = \frac{\mu_0 I a}{2\pi} \ln 2$$

$$\varepsilon = -\frac{\partial \phi}{\partial t} = -\frac{d}{dt} \left(\frac{\mu_0 a I_0 \cos(\omega t)}{2\pi} \ln 2 \right) = \frac{\omega \mu_0 a I_0 \sin(\omega t)}{2\pi} \ln 2$$





The induce current, $I = \frac{\varepsilon}{R} = \frac{\mu_0 a \omega I_0 \sin(\omega t)}{2\pi R} \ln 2$

Therefore, the amplitude of the induced current in the loop is $|I| = \frac{\mu_0 a \omega I_0}{2\pi R} \ln 2$

Correct option is (a)

3. Consider a circular loop of wire of radius r , lying in the xy -plane as shown in figure below. The loop contain a resistor R and a capacitor C , and is placed in a uniform magnetic field which point into the page

and decreases at a rate $\frac{dB}{dt} = -\alpha$, with $\alpha > 0$

(i) Find the maximum amount of charge on the capacitor

(ii) Which plate, a or b , has a higher potential? What causes charges to separate.

Soln. According to Faraday's law,

$$\varepsilon = -\frac{\partial \phi}{\partial t} = -\frac{\partial B}{\partial t} \pi r^2 = \alpha \pi r^2$$

So, we can say capacitor will be charged by the battery

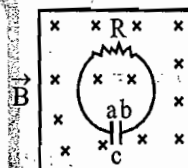
of $\text{emf} = \varepsilon = \alpha \pi r^2$ through the resistance

$$\therefore q = q_0 (1 - e^{-t/RC})$$

$$\therefore q_{\max} = q_0 = \alpha \pi r^2 C$$

According to Lenz's law, the direction of induced emf is such that it tries to oppose the cause of its generation. So, the induce current will flow in clockwise direction. Therefore, electron will flow in anti-clockwise direction.

Therefore, plate b has a lower potential and plate a has a higher potential. Because DC current can not flow through the capacitor.



4. A small magnet is dropped down a long vertical copper tube in a uniform gravitational field. After a long time, the magnet

[JEST 2012]

- (a) Attains a constant velocity (b) moves with a constant acceleration
(c) moves with a constant deceleration (d) executes simple harmonic motion.

Soln. When magnet bar fall down through a long copper tube in a uniform gravitational field, due to Lorentz force an additional force acting on it opposite to its motion due to this it attains a constant velocity.

5. A circular conducting ring of radius ' R ' rotates with constant angular velocity ω about its diameter placed along the x -axis. A uniform magnetic field B is applied along the y -axis. If at time $t=0$ the ring is entirely in the xy -plane, the emf induced in the ring at time $t > 0$ is

[JEST 2012]

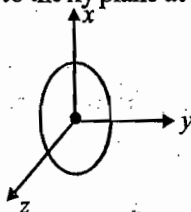
- (a) $B\omega^2 \pi R^2 t$ (b) $B\omega \pi R^2 \tan(\omega t)$ (c) $B\omega \pi R^2 \sin(\omega t)$ (d) $B\omega \pi R^2 \cos(\omega t)$

Soln. Initially the normal of the plate is perpendicular to the xy plane at any time t the normal make the angle ωt

$$\phi = B \cdot ds = B \pi R^2 \sin(\omega t)$$

$$\varepsilon = -\frac{\partial \phi}{\partial t} = \omega \pi R^2 B \cos(\omega t)$$

$$|\varepsilon| = \omega \pi R^2 B \cos(\omega t)$$



Correct option is (d)



6. A large circular coil of N turns and radius R carries a time varying current $i = i_0 \sin(\omega t)$. A small circular coil of n turns and radius r ($r \ll R$) is placed at the center of the large coil such that the coils are concentric and coplanar. The induced emf in the small coil
- (a) Leads the current in the large coil by $\pi/2$
- (b) Lags the current in the large coil by π
- (c) Is in phase with the current in the large coil
- (d) lags the current in the large coil by $\pi/2$

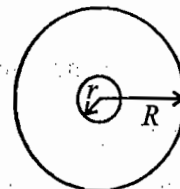
[GATE 2002]

Soln. We have, $I = I_0 \sin(\omega t)$

The magnetic field at the centre of large circular surface is

$$B = \frac{\mu_0 IN}{2R} = \frac{\mu_0 NI_0}{2R} \sin \omega t$$

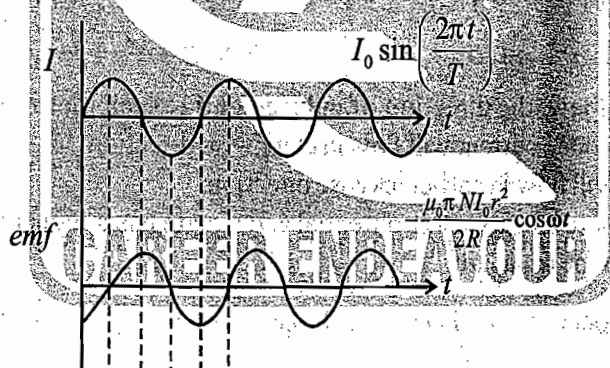
Therefore, the magnetic flux passing through the smaller circular surface.



$$\phi = \frac{\mu_0 NI_0}{2R} \pi r^2 \sin \omega t$$

$$\therefore \text{emf} = -\frac{d\phi}{dt} = -\frac{\mu_0 NI_0}{2R} \pi r^2 \omega \cos \omega t$$

$$\text{emf} = -\frac{\mu_0 \pi N I r^2}{2R} \cos \omega t$$



Therefore, induced emf in the small coil lags behind the current in the large coil by $\frac{\pi}{2}$

Correct option is (d)

7. A conducting loop L of surface area S is moving with a velocity $\vec{v}$ in a magnetic field $\vec{B}(\vec{r}, t) = \vec{B}_0 t^2$, B_0 is a positive constant of suitable dimensions. The emf induced, V_{emf} , in the loop is given by : [GATE 2009]

(a) $-\int_s \frac{\partial \vec{B}}{\partial t} \cdot d\vec{S}$ (b) $\oint_L (\vec{v} \times \vec{B}) \cdot d\vec{L}$ (c) $-\int_s \frac{\partial \vec{B}}{\partial t} \cdot d\vec{S} - \oint_L (\vec{v} \times \vec{B}) \cdot d\vec{L}$ (d) $-\int_s \frac{\partial \vec{B}}{\partial t} \cdot d\vec{S} + \oint_L (\vec{v} \times \vec{B}) \cdot d\vec{L}$

Soln. Let consider an elementary surface is

Therefore, the magnetic flux pass through the elementary area dS is

$$d\phi = \phi \cdot B \cdot dS$$

$$\text{Induced emf} = -\frac{d\phi}{dt} = -\int \frac{\partial \vec{B}}{\partial t} \cdot d\vec{S}$$



PRACTICE SET

- The electric field produced by a uniformly moving charge is
(a) conservative (b) irrotational (c) non-conservative (d) none of these
- A conducting rod of length l rotates with a constant angular velocity ω about its perpendicular bisector. An external uniform magnetic field $\vec{B}$ is applied parallel to the axis of rotation. The pd between the two ends of the rod is
(a) $\frac{1}{2} B\omega l^2$ (b) $\frac{1}{4} B\omega l^2$ (c) $B\omega l^2$ (d) zero
- A small bar magnet is allowed to fall freely from rest along the axis of a very long vertical conducting tube. After sometime the magnet will be found to
(a) come to rest (b) oscillate
(c) move with an acceleration g , (d) move with a constant speed
- The differential form of Faraday's law of electromagnetic induction is
(a) $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$ (b) $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$ (c) $\vec{\nabla} \times \vec{B} = -\frac{\partial \vec{E}}{\partial t}$ (d) $\vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$
- Lenz's law is a consequence of the law of conservation of
(a) charge (b) energy (c) momentum (d) mass
- What are the dimensions of $\nabla \Phi / R$ where $\nabla \Phi$ is the change in magnetic flux and R is the resistance?
(a) $[IT]$ (b) $[MLT^{-3}I^{-1}]$ (c) $[IT^{-1}]$ (d) $[ML^2T^{-2}I^{-1}]$
- A circular loop of radius r is lying in the xy -plane with its centre at the origin. If there exists a time varying magnetic field $\vec{B}(t) = B_0 e^{-\alpha t} \hat{z}$ ($B_0 > 0, \alpha > 0$) then the induced emf in the loop is
(a) $-\pi r^2 \alpha B_0 e^{-\alpha t}$ (b) $-\pi r^2 B_0 e^{-\alpha t}$ (c) $\pi r^2 B_0 e^{-\alpha t}$ (d) $\pi r^2 \alpha B_0 e^{-\alpha t}$
- The current passing through a choke coil of 1 H is decreasing at the rate of 2 A/s . Then, the emf developed in the coil is
(a) 2 V (b) 0.5 V (c) -2 V (d) -0.5 V
- A copper ring is held horizontally and a bar magnet is dropped through the ring with its length along the axis of the ring. Then, the acceleration of the falling magnetic is
(a) equal to g , (b) greater than g (c) less than g (d) zero
- The equation $\vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$ is the differential form of
(a) Faraday's law (b) Gauss's law (c) Ampere's law (d) Coulomb's law

ANSWER KEY

Questions	1	2	3	4	5
Option	(c)	(d)	(d)	(a)	(b)
Questions	6	7	8	9	10
Option	(a)	(d)	(a)	(c)	(c)

Chapter 12

Maxwell's Equation, Poynting vector, EM wave

Displacement current:

Till now we have come across four basic laws about electromagnetism. These are essentially:

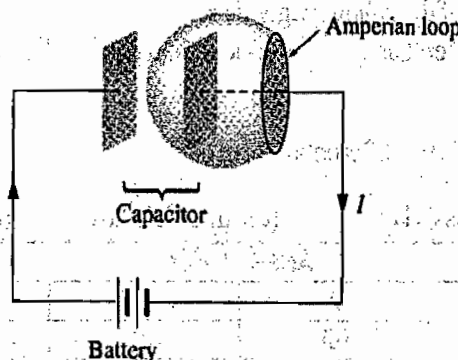
- (i) $\vec{\nabla} \cdot \vec{E} = \frac{1}{\epsilon_0} \rho$ (Gauss's law),
- (ii) $\vec{\nabla} \cdot \vec{B} = 0$ (no name),
- (iii) $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$ (Faraday's law)
- (iv) $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$ (Ampere's law)

Faraday's law (iii) says that changing magnetic field creates a electric field. Does changing electric field produces magnetic field too? Answer is YES. To understand this, consider the following case.

Consider the fig 8.1. There is a capacitor connected with a battery and it is charging. The Ampere's law is:

$$\oint \vec{B} \cdot d\vec{\ell} = \mu_0 I_{enc}$$

If we apply ampere's law in the deep shadowed circular loop right side of right plate of capacitor. The wire, carrying current I , cutting through the loop. So enclosed current by the loop I_{enc} is I . But if we apply the Ampere's law in the balloon shaped loop (lightly shadowed) then no current passing through the loop. So $I_{enc} = \text{Zero!}$ So there must be going something wrong there.



Maxwell fixed the problem by adding a displacement current in the ampere's law as

$$J_d \equiv \epsilon_0 \frac{\partial \vec{E}}{\partial t} \quad \dots (2)$$

Or $I_d \equiv \epsilon_0 \frac{\partial \Phi_E}{\partial t} \quad (3)$



Here Φ_E is Electric flux (flux created by electric field).

So the integral form of Ampere's law becomes

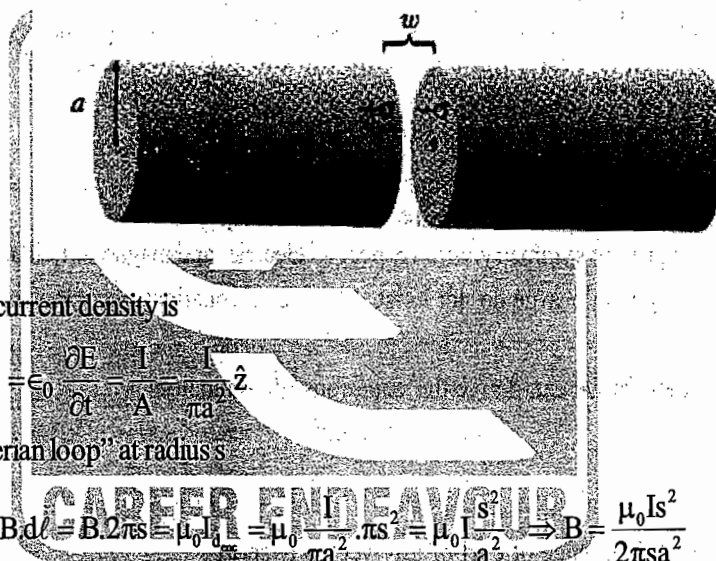
$$\oint \vec{B} \cdot d\vec{\ell} = \mu_0 I_{enc} + \mu_0 \epsilon_0 \int \left(\frac{\partial \vec{E}}{\partial t} \right) \cdot d\vec{a} \quad \dots (4)$$

And in the curl form

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} \quad \dots (5)$$

SOLVED PROBLEMS

1. A fat wire, radius a , carries a constant current I , uniformly distributed over its cross-section. A narrow gap in the wire, of width $w \ll a$, forms a parallel-plate capacitor, as shown in Figure. Find the magnetic field in the gap, at a distance $s < a$ from the axis.



Soln. The displacement current density is

$$J_d = \epsilon_0 \frac{\partial E}{\partial t} = \frac{I}{A} = \frac{I}{\pi a^2} \hat{z}$$

Drawing an "amperian loop" at radius s

$$\oint \vec{B} \cdot d\vec{\ell} = B \cdot 2\pi s = \mu_0 I_d = \mu_0 \frac{I}{\pi a^2} \cdot \pi s^2 = \mu_0 I \frac{s^2}{a^2} \Rightarrow B = \frac{\mu_0 I s^2}{2\pi a^2}$$

$$\vec{B} = \frac{\mu_0 I s}{2\pi a^2} \hat{\phi}$$

Prob: 1

An alternating current $I = I_0 \cos(\omega t)$ flows down a long straight wire, and returns along a co-axial conducting tube of radius a .

(a) What is the Electric field induced? In what direction does the induced electric field point (radial, circumferential, or longitudinal)?

Ans:

$$E(s, t) = \frac{\mu_0 I_0 \omega}{2\pi} \sin(\omega t) \ln\left(\frac{a}{s}\right) \hat{z} \quad \dots (i)$$

Prob: 2: Refer to Prob. 1 to which the correct answer is equation (i)

(b) Find the displacement current density J_d .

(c) Integrate it to get the total displacement current,

$$I_d = \int J_d \cdot d\vec{a}$$

**Maxwell's equations:**

So, the Maxwell's equations now after addition of displacement current are

$$(i) \quad \vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0} \quad (\text{Gauss's law in electrostatic}) \quad \dots (6)$$

$$(ii) \quad \vec{\nabla} \cdot \vec{B} = 0 \quad (\text{Gauss's law in Magnetostatic}) \quad \dots (7)$$

$$(iii) \quad \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad (\text{Faraday's law}) \quad \dots (8)$$

$$(iv) \quad \vec{\nabla} \times \vec{B} = \left. \begin{array}{l} \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} \\ \mu_0 \vec{J} + \mu_0 \frac{\partial \vec{D}}{\partial t} \end{array} \right\} \quad (\text{Ampere's law with Maxwell's correction}) \quad \dots (9)$$

Properties of Maxwell's equation:

- (i) Maxwell's equations are linear
- (ii) Maxwell's equations obey the law of superposition
- (iii) Maxwell's equations are relativistic invariants
- (iv) Maxwell's equations are not symmetry with respect to electric and magnetic fields. Because electric charge exist in nature but magnetic charge do not exist. Maxwell's equations become symmetry when $\rho = 0$ and $J = 0$
- (v) From Maxwell's equations we can derive the equations of continuity

$$\vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$

$$\Rightarrow \quad \vec{\nabla} \cdot (\vec{\nabla} \times \vec{H}) = 0 \quad (\text{divergence of curl of any vector is zero})$$

$$\Rightarrow \quad \vec{\nabla} \cdot \vec{J} + \frac{\partial}{\partial t} (\vec{\nabla} \cdot \vec{D}) = 0$$

$$\Rightarrow \quad \boxed{\vec{\nabla} \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0} \quad \dots (10)$$

Poynting Theorem

We know that the electric energy stored in electromagnetic field is

$$W_e = \frac{\epsilon_0}{2} \int E^2 d\tau$$

And the magnetic energy is

$$W_m = \frac{1}{2\mu_0} \int B^2 d\tau$$

So the total energy stored in the electromagnetic field is

$$U_{em} = \frac{1}{2} \int \left(\epsilon_0 E^2 + \frac{1}{\mu_0} B^2 \right) d\tau \quad \dots (11)$$



Suppose we have some charge and current configuration which, at time 't' produces fields E and B . In the next instant, dt , the charges move around a bit. Question: How much work dW , is done by the electromagnetic forces acting on these charges in the interval dt ? According to the Lorentz law, the work done on a charge q is

$$F \cdot dl = q(E + v \times B) \cdot v dt = qE \cdot v dt$$

Now, $q = \rho d\tau$ and $\rho v = J$, so the rate at which work is done on all the charges in a volume v is:

$$\frac{dW}{dt} = \int_v (E \cdot J) d\tau$$

This quantity (dw/dt) can be proved to be equals to

$$\frac{dW}{dt} = -\frac{d}{dt} \int_v \frac{1}{2} \left(\epsilon_0 E^2 + \frac{1}{\mu_0} B^2 \right) d\tau - \frac{1}{\mu_0} \oint_s (E \times B) \cdot da \quad \dots (12)$$

where 'S' is the surface bounding v . This is Poynting's theorem; it is the "work-energy theorem" of electrodynamics. The first integral on the right is the total energy stored in the fields, U_{em} . The second term evidently represents the rate at which energy is carried out of v , across its boundary surface, by the electromagnetic fields. Poynting's theorem says, then, that the work done on the charges by the electromagnetic force is equal to the decrease in energy stored in the field, less the energy that flowed out through the surface.

The energy per unit time, per unit area, transported by the fields is called the Poynting vector:

$$\vec{S} = \frac{1}{\mu_0} (\vec{E} \times \vec{B}) \quad \dots (13)$$

By the definition of S , dw/dt can be written as

$$\frac{dW}{dt} = \frac{dU_{em}}{dt} - \oint_s \vec{S} \cdot d\vec{a} \quad \dots (14)$$

Of course, the work W done on the charges will increase their mechanical energy (kinetic, potential, or whatever).

If we let U_{mech} denote the mechanical energy density, so that

$$\frac{dW}{dt} = \frac{d}{dt} \int_v U_{mech} d\tau$$

And use U_{em} for the energy density of the fields,

$$u_{em} = \frac{1}{2} \left(\epsilon_0 E^2 + \frac{1}{\mu_0} B^2 \right)$$

Then the equation becomes

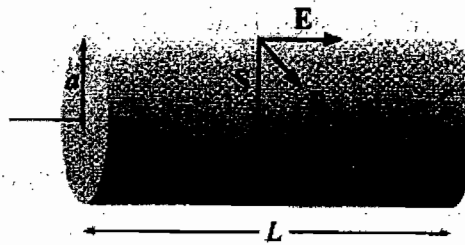
$$\frac{d}{dt} \int_v (u_{mech} + u_{em}) d\tau = -\oint_s \vec{S} \cdot d\vec{a} = -\int_v (\vec{\nabla} \cdot \vec{S}) d\tau$$

So, in more compact form

$$\frac{\partial}{\partial t} (u_{mech} + u_{em}) = -\vec{\nabla} \cdot \vec{S} \quad \dots (15)$$

2.

When current flows down a wire, work is done, which shows up as Joule heating of the wire. Though there are certainly easier ways to do it, the energy per unit time delivered to the wire can be calculated using the Poynting vector. Assuming it's uniform, the electric field parallel to the wire is



$$E = \frac{V}{L}$$

where V is the potential difference between the ends and L is the length of the wire (Fig. 8.3). The magnetic field is "circumferential"; at the surface (radius a) it has the value

$$B = \frac{\mu_0 I}{2\pi a}$$

The magnitude of the Poynting vector is

$$S = \frac{1}{\mu_0} \frac{V}{L} \frac{\mu_0 I}{2\pi a} = \frac{VI}{2\pi aL}$$

And it points radially inward. The energy per unit time passing in through the surface of the wire is therefore,

$$\int S \cdot da = S(2\pi aL) = VI$$

Electromagnetic wave in free space:

In free space, $\rho = 0$ and $J = 0$

The Maxwell's equations take the following form

$$\vec{\nabla} \cdot \vec{E} = 0 \quad \dots (16)$$

$$\vec{\nabla} \cdot \vec{H} = 0 \quad \dots (17)$$

$$\vec{\nabla} \times \vec{E} = -\mu_0 \frac{\partial \vec{H}}{\partial t} \quad \dots (18)$$

$$\vec{\nabla} \times \vec{H} = \epsilon_0 \frac{\partial \vec{E}}{\partial t} \quad \dots (19)$$

Taking curl of the equation (8.4-4), we get,

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{H}) = \epsilon_0 \frac{\partial}{\partial t} (\vec{\nabla} \times \vec{E})$$

$$\Rightarrow \vec{\nabla} (\vec{\nabla} \cdot \vec{H}) - \vec{\nabla}^2 \vec{H} = -\mu_0 \epsilon_0 \frac{\partial^2 \vec{H}}{\partial t^2} \quad [\text{Using equation (18 and 17)}]$$

$$\Rightarrow \boxed{\vec{\nabla}^2 \vec{H} - \mu_0 \epsilon_0 \frac{\partial^2 \vec{H}}{\partial t^2} = 0} \quad \dots (20)$$

Similarly taking used of equation (18) and using equation (16) and (19), we get

$$\boxed{\vec{\nabla}^2 \vec{E} - \mu_0 \epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2} = 0} \quad \dots (21)$$

$$\nabla^2 \psi - \frac{1}{c^2} \frac{\partial^2 \psi}{\partial t^2} = 0 \quad \dots (22)$$

So, from equations (17), (21) and (20) we can write the speed of electromagnetic wave propagation in free space is

$$c = \frac{1}{\sqrt{\epsilon_0 \mu_0}} \approx 3 \times 10^8 \text{ m/sec}$$

which is speed of light in free space. This says that the light is an electromagnetic wave.

Plane Wave solutions:

The solution of the equations (5) and (6) are plane wave.

$$\left. \begin{aligned} \vec{E}(r, t) &= \vec{E}_0 e^{i(\vec{k} \cdot \vec{r} - \omega t)} \\ \vec{H}(r, t) &= \vec{H}_0 e^{i(\vec{k} \cdot \vec{r} - \omega t)} \end{aligned} \right\} \quad \dots (23)$$

$\vec{E}_0, \vec{H}_0$ are the complex amplitude, and $\vec{k}$ is the propagating constant. $\vec{k}$ is defined as

$$k = \frac{2\pi}{\lambda} \hat{k} = \frac{\omega}{c} \hat{k} \quad (\hat{k} \text{ is unit vector along propagation direction})$$

Relative direction of $\vec{E}, \vec{H}, \vec{k}$

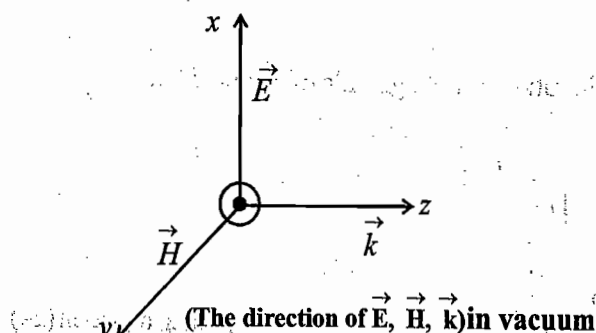
We have solution of wave equation,

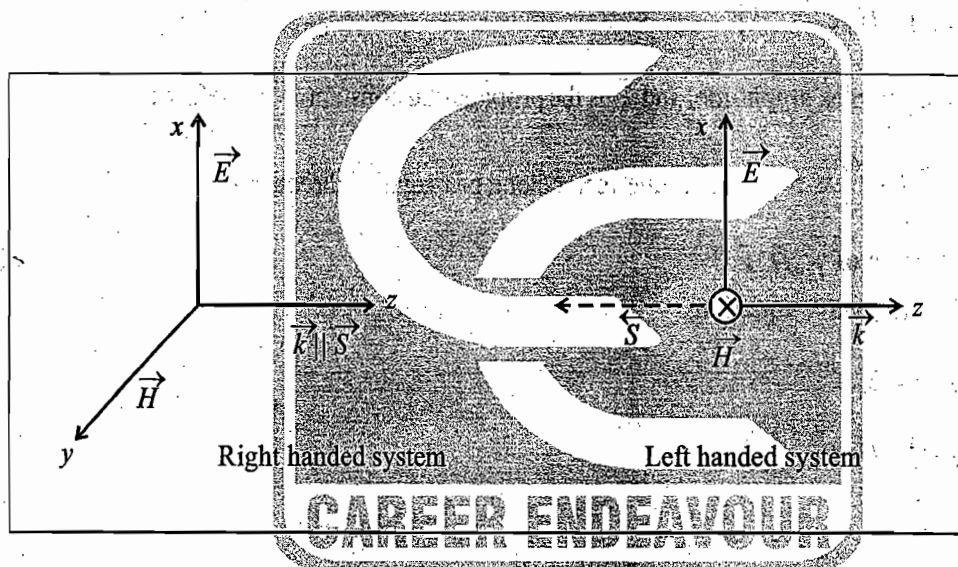
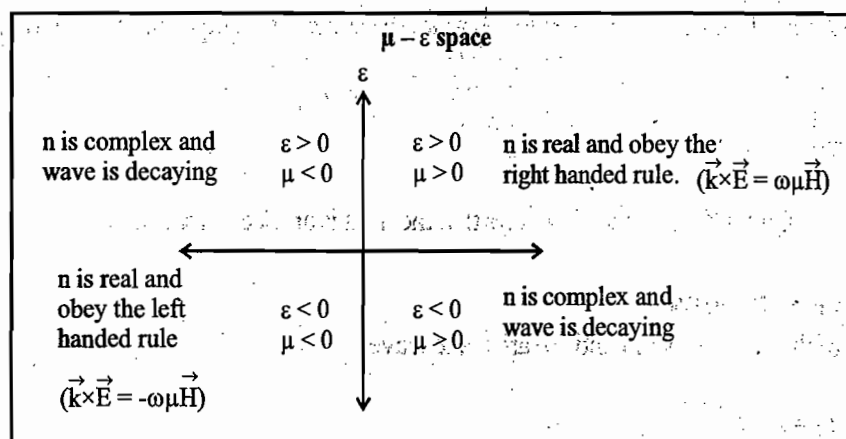
$$\left. \begin{aligned} \vec{E}(r, t) &= \vec{E}_0 e^{i(\vec{k} \cdot \vec{r} - \omega t)} \\ \vec{H}(r, t) &= \vec{H}_0 e^{i(\vec{k} \cdot \vec{r} - \omega t)} \end{aligned} \right\}$$

Now,

$$\left. \begin{aligned} \vec{\nabla} \cdot \vec{E} &= 0 \Rightarrow i\vec{k} \cdot \vec{E} = 0 \quad \therefore \vec{k} \perp \vec{E} \\ \vec{\nabla} \cdot \vec{H} &= 0 \Rightarrow i\vec{k} \cdot \vec{H} = 0 \quad \therefore \vec{k} \perp \vec{H} \\ \vec{\nabla} \times \vec{E} &= -\frac{\partial \vec{B}}{\partial t} \Rightarrow i\vec{k} \times \vec{E} = i\omega\mu_0 \vec{H} \quad \therefore \vec{k} \times \vec{E} = \omega\mu_0 \vec{H} \\ \vec{\nabla} \times \vec{H} &= \epsilon_0 \frac{\partial \vec{E}}{\partial t} \Rightarrow i\vec{k} \times \vec{H} = -i\omega\epsilon_0 \vec{E} \quad \therefore \vec{k} \times \vec{H} = \omega\epsilon_0 \vec{E} \end{aligned} \right\} \quad \dots (24)$$

From equation we can say $\vec{k}, \vec{E}, \vec{H}$ are mutually perpendicular to each other.



**Special Note :****Right handed and Left handed systems :****Relative phase of $\vec{E}$ and $\vec{H}$:**

We have the wave equation $\nabla^2 E - \epsilon_0 \mu_0 \frac{\partial^2 E}{\partial t^2} = 0$ and the solution, $\vec{E} = \vec{E}_0 e^{i(\vec{k} \cdot \vec{r} - \omega t)}$

So, we can write,

$$k^2 - \epsilon_0 \mu_0 \omega^2 = 0$$

$$k^2 = \epsilon_0 \mu_0 \omega^2 \quad \dots (25)$$

Therefore, k is real quantity in free space and $\vec{E}$ and $\vec{H}$ are in phase.

Wave Impedance :

Let z_0 is a quantity of the ratio of the magnitude of $\vec{E}$ and $\vec{H}$ is

$$[z_0] = \frac{[VL^{-1}]}{[IL^{-1}]} = \frac{[V]}{[I]} = [R]$$

$$z_0 = \frac{|\vec{E}|}{\vec{r}} = \frac{\mu_0 \omega}{1} = \sqrt{\frac{\mu_0}{\epsilon_0}} \quad [\text{Using equation (24)}]$$



$$\mu_0 = 4\pi \times 10^{-7} \quad \text{S.I. unit}$$

$$\epsilon_0 = 8.85 \times 10^{-12} \quad \text{S.I. unit}$$

$$z_0 = 376.6 \Omega$$

z_0 is called the impedance of wave in free space.

Poynting vector :

The Poynting's vector for the plane electromagnetic wave in free space is

$$\begin{aligned} \vec{s} &= \vec{E} \times \vec{H} = \frac{1}{\mu_0 \omega} \vec{E} \times (\vec{k} \times \vec{E}) \quad [\text{Using equation (24)}] \\ &= \frac{1}{\mu_0 \omega} [\vec{k} (\vec{E} \cdot \vec{E}) - \vec{E} (\vec{E} \cdot \vec{k})] = \frac{E^2}{\mu_0 \omega} \vec{k} \quad \dots (26) \end{aligned}$$

Thus, the energy flow is in the direction of wave propagation. Since $\vec{E}$ is normal to $\vec{k}$, from equation (9), we can write in terms of magnitude,

$$kE = \mu_0 \omega H \quad \text{or} \quad \sqrt{\epsilon_0} E = \sqrt{\mu_0} H \quad \text{or} \quad \frac{1}{2} \epsilon_0 E^2 = \frac{1}{2} \mu_0 H^2 \quad \dots (28)$$

This shows that in case of electromagnetic waves in free space electromagnetic energy is equally shared between electric and magnetic fields.

Total electromagnetic energy density is

$$u = \frac{1}{2} \epsilon_0 E^2 + \frac{1}{2} \mu_0 H^2 = \epsilon_0 E^2 \quad \dots (29)$$

So, equation (11) can also be written as

$$\vec{s} = uc\hat{n} \quad \dots (14)$$

where, $c = \frac{1}{\sqrt{\mu_0 \epsilon_0}} = \frac{\omega}{k}$ is the speed of the wave.

Time average Poynting's vector:

$$\langle \vec{s} \rangle = \frac{1}{2} \text{Re} [\vec{E}^* \times \vec{H}]$$

The magnitude of the time average of the Poynting's vector is called the intensity of radiation (I). Thus, the intensity,

$$I = \langle s \rangle = \frac{1}{2} E_0 H_0 = \frac{E_0}{\sqrt{2}} \frac{H_0}{\sqrt{2}} = E_{rms} \times H_{rms}$$

$$\text{We also know, } \langle \mu \rangle = \langle \epsilon_0 E_0^2 \rangle = \frac{1}{2} \epsilon_0 E_0^2 = \epsilon_0 E_{rms}^2$$

Since, Maxwell's equation is symmetric of electric field and magnetic field.

So, we can write,

$$\sqrt{\epsilon_0} E_{rms} = \sqrt{\mu_0} H_{rms}$$



$$\therefore \frac{I}{\langle u \rangle} = \frac{1}{\epsilon_0} \cdot \frac{H_{rms}}{E_{rms}} = \frac{1}{\sqrt{\epsilon_0 \mu_0}} = c$$

$$\text{or } I = \langle u \rangle c \quad \dots (30)$$

Plane Electromagnetic Waves in an isotropic Dielectric Medium:

Maxwell's equation for a Homogeneous, non-conducting dielectric are

$$\left. \begin{aligned} \vec{\nabla} \cdot \vec{E} &= 0 \\ \vec{\nabla} \cdot \vec{H} &= 0 \\ \vec{\nabla} \times \vec{E} &= -\mu \frac{\partial \vec{H}}{\partial t} \\ \vec{\nabla} \times \vec{H} &= \epsilon \frac{\partial \vec{E}}{\partial t} \end{aligned} \right\} \quad \dots (31)$$

All the quantities will be same like wave in free space only ϵ_0 replace by ϵ .

Electromagnetic waves in Plasma :

Plasma is quasineutral gas of charged and neutral particles which exhibits collective behaviour. Here, we neglect the motion of ions because it is massive as compared to the electrons.

Let $\vec{E} = \vec{E}(r)e^{-i\omega t}$ is the incident field.

Therefore, the equation motion of electron under the action of the incident electromagnetic field is

$$m \frac{d\vec{v}}{dt} = e\vec{E} \quad \dots (32)$$

$$\therefore \vec{v} = \frac{e\vec{E}(r)}{-i\omega m} = \frac{ie\vec{E}(r)}{m\omega}$$

If n_0 be the electron density inside the plasma

$$\vec{J} = n_0 e \vec{v} = \frac{in_0 e^2}{m\omega} \vec{E}(r) = \sigma \vec{E} \quad \text{where, } \sigma = \frac{in_0 e^2}{m\omega}$$

The Maxwell's equation in plasma,

$$\left. \begin{aligned} \vec{\nabla} \cdot \vec{E} &= \frac{(n_i - n_e)e}{\epsilon_0} = 0 \quad (n_i = n_e = n_0) \\ \vec{\nabla} \cdot \vec{H} &= 0 \\ \vec{\nabla} \times \vec{E} &= -\mu_0 \frac{\partial \vec{H}}{\partial t} \\ \vec{\nabla} \times \vec{H} &= \vec{J} + \epsilon_0 \frac{\partial \vec{E}}{\partial t} \end{aligned} \right\} \quad \dots (33)$$

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = -\mu_0 \left(\vec{\nabla} \times \frac{\partial \vec{H}}{\partial t} \right)$$

$$\Rightarrow \nabla^2 E - \mu_0 \frac{\partial \bar{J}}{\partial t} - \mu_0 \epsilon_0 \frac{\partial^2 E}{\partial t^2} = 0$$

$$\Rightarrow \nabla^2 E - \mu_0 \sigma \frac{\partial E}{\partial t} - \epsilon_0 \mu_0 \frac{\partial^2 E}{\partial t^2} = 0 \quad \dots (34)$$

Let plane wave solution of the equation (34) is

$$E(r, t) = \vec{E}_0 e^{i(\vec{k} \cdot \vec{r} - \omega t)}$$

Putting this value in equation (19), we get

$$-k^2 + i\omega\sigma\mu_0 + \omega^2\epsilon_0\mu_0$$

Put, $\sigma = \frac{in_0 e^2}{m\omega^2}$

$$\Rightarrow -k^2 - \omega \frac{n_0 e^2}{m\omega^2} \mu_0 + \omega^2 \epsilon_0 \mu_0 = 0$$

$$\Rightarrow k^2 = \frac{\omega^2}{c^2} - \frac{n_0 e^2}{\epsilon_0 m c^2} \quad \left[\text{let } \omega_p^2 = \frac{n_0 e^2}{\epsilon_0 m} \right]$$

$$\Rightarrow k^2 c^2 = \omega^2 - \omega_p^2$$

It is called dispersion relation.

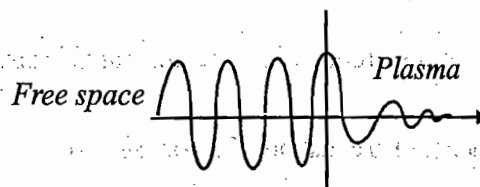
$$\omega_p = \sqrt{\frac{n_0 e^2}{\epsilon_0 m}} = 18\pi \sqrt{n_0} \quad \text{is called electron plasma frequency.}$$



Special Case:

(1) If $\omega > \omega_p$ then k is real and wave can propagate through this medium.

(2) If $\omega < \omega_p$ then k becomes complex and amplitude of the field falls off exponentially and cannot pass through it.



$$\frac{1}{k} = \delta = \frac{c}{\sqrt{\omega_p^2 - \omega^2}} \quad \text{is the penetration depth.}$$

(3) Phase velocity, $v_\phi = \frac{\omega}{k} \quad \dots (35)$

(4) The group velocity of the wave is $v_g = \frac{d\omega}{dk} = \frac{kc^2}{\omega} = \frac{c^2}{v_\phi} \quad \dots (36)$

From equation (35) and (36), we get, $v_g v_\phi = c^2$



Plane Electromagnetic Waves in a Conducting Medium:

Consider a linear, homogeneous and isotropic conducting medium having permittivity ϵ , permeability μ , conductivity σ and the medium is charge free ($\rho = 0$) and external-current free such that the currents existing in the medium are induced only by the electromagnetic wave itself. Thus, we take $\vec{J} = \sigma \vec{E}$. Any initial charge distribution within the conductor dies out quickly. The charges move to the surface and make $\rho = 0$ inside. The Maxwell's equations are

$$(i) \vec{\nabla} \cdot \vec{E} = 0 \quad (37.a)$$

$$(ii) \vec{\nabla} \cdot \vec{H} = 0 \quad (37.b)$$

$$(iii) \vec{\nabla} \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t} \quad (37.c)$$

$$(iv) \vec{\nabla} \times \vec{H} = \sigma \vec{E} + \epsilon \frac{\partial \vec{E}}{\partial t} \quad (37.d)$$

Taking curl on equation (37.c) and using equation (37.d), we get

$$\therefore \vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = -\mu \left(\vec{\nabla} \times \frac{\partial \vec{H}}{\partial t} \right)$$

$$\text{Or, } \vec{\nabla} (\vec{\nabla} \cdot \vec{E}) - \vec{\nabla}^2 \vec{E} = -\mu \frac{\partial}{\partial t} (\vec{\nabla} \times \vec{H}) = -\sigma \mu \frac{\partial \vec{E}}{\partial t} - \epsilon \mu \frac{\partial^2 \vec{E}}{\partial t^2}$$

Now using equation (i), we get

$$\vec{\nabla}^2 \vec{E} - \sigma \mu \frac{\partial \vec{E}}{\partial t} - \epsilon \mu \frac{\partial^2 \vec{E}}{\partial t^2} = 0 \quad \dots (37.e)$$

Similarly, taking curl of equation (37.d) and using equation (37.c) and (37.a) we get

$$\vec{\nabla}^2 \vec{H} - \sigma \mu \frac{\partial \vec{H}}{\partial t} - \epsilon \mu \frac{\partial^2 \vec{H}}{\partial t^2} = 0 \quad \dots (38)$$

Plane wave solutions,

$$\vec{E}(\vec{r}, t) = \vec{E}_0 e^{j(\vec{k} \cdot \vec{r} - \omega t)} \text{ and } \vec{H}(\vec{r}, t) = \vec{H}_0 e^{j(\vec{k} \cdot \vec{r} - \omega t)} \quad \dots (39)$$

where $\vec{E}_0$ and $\vec{H}_0$ are complex amplitudes which are constants in space time, $\vec{k} = k\hat{n}$ is the propagation vector.

Substituting the solution (39) in equation (37.e) or (38), we get

$$[-k^2 + j\omega\sigma\mu + \omega^2\epsilon\mu] \vec{E} = 0$$

For non-zero solution,

$$k^2 = \omega^2\epsilon\mu + j\omega\sigma\mu \quad \dots (40)$$

k is thus, complex in character here. Let $k = \alpha + j\beta$. Then

$$k^2 = \alpha^2 - \beta^2 + j2\alpha\beta \quad \dots (41)$$

Comparing equation (viii) and (ix), we get

$$\alpha^2 - \beta^2 = \omega^2\epsilon\mu \text{ and } 2\alpha\beta = \omega\sigma\mu$$

On solving these two equations, we get



$$\alpha = \omega \sqrt{\frac{\epsilon \mu}{2}} \left[\sqrt{1 + \left(\frac{\sigma}{\omega \epsilon} \right)^2} + 1 \right]^{1/2} \quad \dots (42.a)$$

$$\beta = \omega \sqrt{\frac{\epsilon \mu}{2}} \left[\sqrt{1 + \left(\frac{\sigma}{\omega \epsilon} \right)^2} - 1 \right]^{1/2} \quad \dots (42.b)$$

$$\therefore \vec{E}(\vec{r}, t) = \vec{E}_0 \cdot e^{-\beta \hat{n} \cdot \vec{r}} \cdot e^{j(\alpha \hat{n} \cdot \vec{r} - \omega t)}$$

$$\text{And } \vec{H}(\vec{r}, t) = \vec{H}_0 \cdot e^{-\beta \hat{n} \cdot \vec{r}} \cdot e^{j(\alpha \hat{n} \cdot \vec{r} - \omega t)} \quad \dots (43)$$

These solutions show that the field amplitudes are spatially attenuated. The physical reason of the attenuation is that the wave sets up electric current in the medium, which causes dissipation of energy in the form of Joule heating. The quantity β , the imaginary part of wave number k , is a measure of attenuation and is called attenuation constant. It depends on the frequency ω and conductivity σ .

Case-I : For non-conducting medium $\sigma = 0 \cdot \beta = 0$

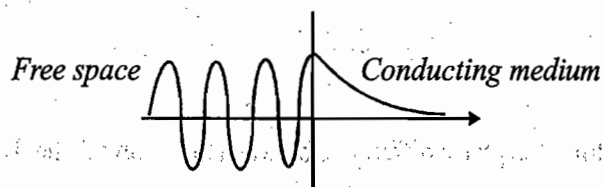
There is no attenuation.

Case-II: For conducting medium $\frac{\sigma}{\epsilon \omega} \gg 1$

$$\beta = \omega \sqrt{\frac{\mu \epsilon}{2}} \left[\sqrt{1 + \left(\frac{\sigma}{\omega \epsilon} \right)^2} - 1 \right]^{1/2} = \omega \sqrt{\frac{\mu \epsilon}{2}} \left(\sqrt{\frac{\sigma}{\omega \epsilon}} \right) = \sqrt{\frac{\omega \sigma \mu}{2}}$$

The quantity $\frac{1}{\beta}$ measures the depth at which electromagnetic wave entering a conductor is attenuated to $\frac{1}{e}$ of its initial amplitude at the surface. It is known as skin depth or penetration depth into a conducting medium. Thus, skin depth δ for a good conductor is

$$\delta = \frac{1}{\beta} = \sqrt{\frac{2}{\omega \sigma \mu}} \quad \dots (44)$$



Wavelength, propagation speed and the index of refraction:

The real part of k , i.e. α determines the wavelength, the propagation speed of the wave and the index of refraction of the conductor, in the usual way. Thus,

$$\lambda_c = \frac{2\pi}{\alpha}, v = \frac{\omega}{\alpha} \text{ and } n = \frac{c\alpha}{\omega}$$

For a good conductor, $\alpha = \sqrt{\frac{\omega \sigma \mu}{2}}$ and hence,

$$\lambda_c = \frac{2\pi}{\alpha} = \frac{2\pi}{\sqrt{\frac{\omega \sigma \mu}{2}}} = \frac{2\pi \sqrt{2}}{\sqrt{\omega \sigma \mu}}$$



The Skin depth $\delta = \frac{\lambda_c}{2\pi}$

Relative directions of $\vec{E}$, $\vec{H}$ and $\vec{k}$:

Substituting the solutions (39) in equation (37.a) and (37.b), we get

$$\vec{k} \cdot \vec{E} = 0 \text{ and } \vec{k} \cdot \vec{H} = 0$$

These equations indicate that $\vec{E}$ and $\vec{H}$ are both perpendicular to the direction of propagation. So, electromagnetic waves in a conducting medium are transverse in nature.

Again, the substitution of the solutions (39) in equation (37.c) and (37.d) gives

$$j\vec{k} \times \vec{E} = -\mu(-j\omega\vec{H}) \text{ or } \boxed{\vec{k} \times \vec{E} = \mu\omega\vec{H}} \quad \dots (45)$$

$$\text{And } j\vec{k} \times \vec{H} = \sigma\vec{E} = \sigma\vec{E} - j\omega\epsilon\vec{E} \text{ or } \boxed{\vec{k} \times \vec{H} = -(\omega\epsilon + j\sigma)\vec{E}} \quad \dots (46)$$

Relative Phase of $\vec{E}$ and $\vec{H}$:

From equation (45), we have

$$\vec{H} = \frac{1}{\mu\omega} (\vec{k} \times \vec{E}) = \frac{k}{\mu\omega} (\hat{n} \times \vec{E}) = \frac{\alpha + j\beta}{\mu\omega} (\hat{n} \times \vec{E}) \quad \dots (47)$$

This equation shows that $\vec{E}$ and $\vec{H}$ are not in phase in a conductor.

Writing $\alpha + j\beta = \sqrt{\alpha^2 + \beta^2} e^{j\phi}$, $\phi = \tan^{-1}\left(\frac{\beta}{\alpha}\right)$ and using equation (47) may be rewritten as

$$\vec{H} = \frac{\sqrt{\alpha^2 + \beta^2}}{\mu\omega} (\hat{n} \times \vec{E}_0) e^{j(\omega t - \alpha z - \phi)} \quad \dots (48)$$

Where,

$$\sqrt{\alpha^2 + \beta^2} = \omega\sqrt{\epsilon\mu} \left[1 + \left(\frac{\sigma}{\omega\epsilon} \right)^2 \right]^{1/4}$$

Thus, $\vec{H}$ lags behind $\vec{E}$ in time by the phase angle

$$\phi = \tan^{-1} \frac{\beta}{\alpha} = \frac{1}{2} \tan^{-1} \left(\frac{\sigma}{\omega\epsilon} \right)$$

For good conductors $\alpha \approx \beta$ and $\phi = 45^\circ$. Therefore, the phase difference between the $\vec{E}$ and $\vec{H}$ fields in a perfect conductor is 45° .

$$\left| \frac{\vec{H}}{\vec{E}} \right| = \frac{H_0}{E_0} = \frac{\sqrt{\alpha^2 + \beta^2}}{\mu\omega} = \sqrt{\frac{\epsilon}{\mu}} \left[1 + \left(\frac{\sigma}{\omega\epsilon} \right)^2 \right]^{1/4}$$

$$\text{For good conductors, } \left| \frac{\vec{H}}{\vec{E}} \right| = \sqrt{\frac{\sigma}{\omega\mu}}$$

Thus, in this case, $|\vec{H}| \gg |\vec{E}|$

which indicates that in a good conducting medium the field energy is not equally shared between $\vec{E}$ and $\vec{H}$



Poynting's Vector:

The time average Poynting vector is

$$\begin{aligned}\langle \vec{s} \rangle &= \frac{1}{2} \text{Re}(\vec{E} \times \vec{H}^*) \\ \Rightarrow \langle \vec{s} \rangle &= \frac{1}{2} \frac{\sqrt{\alpha^2 + \beta^2}}{\mu\omega} \cdot \text{Re}[\vec{E} \times (\hat{n} \times \vec{E}^*)] e^{-j\phi} \\ &= \frac{\sqrt{\alpha^2 + \beta^2}}{2\mu\omega} \text{Re}[\hat{n}(\vec{E} \cdot \vec{E}^*) - \vec{E}^*(\vec{E} \cdot \hat{n})] e^{-j\phi}\end{aligned}$$

Now putting $\vec{E} \cdot \hat{n} = 0$ and using (43), we get

$$\langle \vec{s} \rangle = \frac{\sqrt{\alpha^2 + \beta^2}}{2\mu\omega} \cdot E_0^2 e^{-2\beta\hat{n}\cdot\vec{r}} \cdot \cos\phi \cdot \hat{n} \quad \dots (49)$$

For good conductors $\alpha \approx \beta \approx \sqrt{\frac{\omega\sigma\mu}{2}}$ and $\phi = 45^\circ$. Then

$$\langle \vec{s} \rangle = \frac{1}{2} \sqrt{\frac{\sigma}{2\mu\omega}} E_0^2 e^{-2\beta\hat{n}\cdot\vec{r}} \hat{n} \quad \dots (50)$$

Thus, energy flow is along the direction of propagation of the wave and is damped exponentially.

Average electric energy density is

$$\langle u_e \rangle = \frac{1}{2} \text{Re} \left(\frac{1}{2} (\vec{E} \cdot \vec{D}^*) \right) = \frac{1}{4} \epsilon \text{Re}(\vec{E} \cdot \vec{E}^*) = \frac{1}{4} \epsilon E_0^2 e^{-2\beta\hat{n}\cdot\vec{r}} \quad \dots (51)$$

Average electric energy density is

$$\langle u_m \rangle = \frac{1}{2} \text{Re} \left(\frac{1}{2} (\vec{B} \cdot \vec{H}^*) \right) = \frac{1}{4} \mu \text{Re}(\vec{H} \cdot \vec{H}^*) = \frac{1}{4} \mu H_0^2 e^{-2\beta\hat{n}\cdot\vec{r}} \quad \dots (52)$$

Thus,
$$\frac{\langle u_m \rangle}{\langle u_e \rangle} = \frac{\mu H_0^2}{\epsilon E_0^2} = \left[1 + \left(\frac{\sigma}{\omega\epsilon} \right)^2 \right] \quad \dots (53)$$

For good conductors $\langle u_m \rangle \gg \langle u_e \rangle$ i.e., the field energy inside is almost entirely magnetic in nature.

Total time average energy density is

$$\langle u \rangle = \langle u_e \rangle + \langle u_m \rangle = \frac{1}{4} e^{-2\beta\hat{n}\cdot\vec{r}} [\epsilon E_0^2 + \mu H_0^2] \quad \dots (54)$$

Using equation (53) we can write,

$$\langle u \rangle = \frac{1}{4} e^{-2\beta\hat{n}\cdot\vec{r}} \cdot \epsilon E_0^2 \left[1 + \sqrt{1 + \left(\frac{\sigma}{\epsilon\omega} \right)^2} \right]$$

In term of α , the real part of wave number as given by (42.a), it can be expressed as

$$\langle u \rangle = \frac{\alpha^2}{2\mu\omega^2} E_0^2 e^{-2\beta\hat{n}\cdot\vec{r}}$$

Electromagnetic Waves inside an anisotropic linear medium

Inside an anisotropic linear medium electric permittivity is a tensor quantity and $\vec{D}$ is no more parallel to $\vec{E}$.

In an anisotropic linear medium we can write



$$\Rightarrow \begin{pmatrix} D_x \\ D_y \\ D_z \end{pmatrix} = \begin{bmatrix} \epsilon_{xx} & \epsilon_{xy} & \epsilon_{xz} \\ \epsilon_{yx} & \epsilon_{yy} & \epsilon_{yz} \\ \epsilon_{zx} & \epsilon_{zy} & \epsilon_{zz} \end{bmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}$$

The Maxwell's equations are

$$1. \quad \vec{\nabla} \cdot \vec{D} = \rho_f \Rightarrow \boxed{\vec{\nabla} \cdot \vec{D} = 0} \quad \because \rho_f = 0 \text{ for dielectric medium.} \quad \dots (56)$$

But $\boxed{\vec{\nabla} \cdot \vec{E} \neq 0}$ because it is anisotropic.

$$2. \quad \vec{\nabla} \cdot \vec{B} = 0 \quad \dots (57)$$

$$3. \quad \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad \dots (58)$$

$$4. \quad \vec{\nabla} \times \vec{H} = \vec{J}_f + \frac{\partial \vec{D}}{\partial t} \Rightarrow \vec{\nabla} \times \vec{H} = \frac{\partial \vec{D}}{\partial t} \quad [\vec{J}_f = 0] \quad \dots (59)$$

From equation (56) and (57) we get

$$\vec{k} \cdot \vec{D} = 0 \quad \vec{k} \cdot \vec{B} = 0$$

Therefore, k , D and B are mutually perpendicular with each other.

Since, $\vec{D}$ is no more parallel to $\vec{E}$ and the poynting vector.

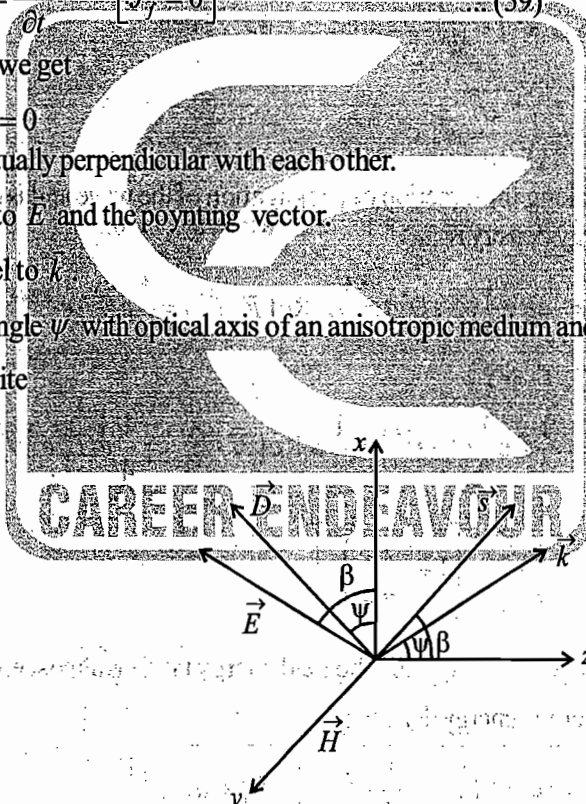
$\vec{S} = \vec{E} \times \vec{H}$ is no more parallel to k .

Suppose $\vec{k}$ vector make an angle ψ with optical axis of an anisotropic medium and $\vec{S}$ vector make an angle with z -axis β then we can write

$$\boxed{\frac{D_z}{D_x} = -\tan \psi} \quad \dots (60)$$

$$\boxed{\frac{E_z}{E_x} = -\frac{\epsilon_x}{\epsilon_z} \tan \psi} \quad \dots (61)$$

$$\boxed{\frac{E_z}{E_x} = -\tan \beta} \quad \dots (62)$$



Case- (i) If $\epsilon_x > \epsilon_z$ then $\beta > \psi$

(ii) If $\epsilon_x < \epsilon_z$ then $\beta < \psi$

Maxwell's equations for static fields :

$$(1) \quad \vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0} \quad (2) \quad \vec{\nabla} \cdot \vec{B} = 0 \quad (3) \quad \vec{\nabla} \times \vec{E} = 0 \quad (4) \quad \vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$$

Polarization of Electromagnetic Wave:

Polarization direction:

Direction of polarisation vector is the direction of electric field.

**Unpolarized light :**

When an ordinary light propagating its electric field is vibrating in the plane perpendicular to the propagation direction.

Plane of vibration :

This plane contains the E -vector and the propagation vector $\vec{k}$ both.

Plane of Polarisation :

The plane perpendicular to the plane of vibration is called polarisation plane it contains only the $\vec{E}$ -vector.

Circular polarised light:

If the $\vec{E}$ -vector trace out a circle at a fixed space such light is called circular polarised light.

Right circular polarised light :

If on looking towards the incoming light the resultant light vector appears to rotate clockwise, then the light is called right circular polarised light.

Elliptically polarised :

If the $\vec{E}$ -vector trace out an elliptical path then it is called elliptically polarised light.

Production of plane polarised light :

- (i) By polarised method
- (ii) By refraction method
- (iii) By plane retardation
- (iv) Beam splitter
- (v) By Scattering

Polarisation by reflection and Brewster's law:

If an ordinary beam of light is incident on reflecting surface at a particular angle, it is found that the reflected light beam becomes almost wholly a plane polarised light, the plane of vibration being perpendicular to the plane of incidence. This particular angle is called angle of polarisation (i_p).

Brewster discovered that at the polarised angle the reflected and refracted rays are just 90° apart. So, the refracting angle

$$r = 90^\circ - i_p$$

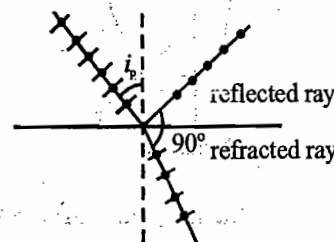
$$\therefore n_1 \sin i_p = n_2 \sin r$$

$$\Rightarrow n_1 \sin i_p = n_2 \sin (90^\circ - i_p)$$

$$\Rightarrow n_1 \sin i_p = n_2 \cos i_p$$

$$\Rightarrow \tan i_p = \left(\frac{n_2}{n_1} \right)$$

$$\Rightarrow \boxed{\tan i_p = n_2} \quad (\text{for air } n_1 = 1) \quad \dots (63)$$

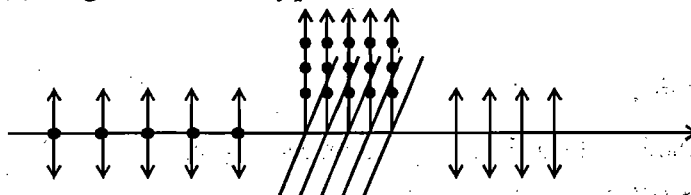


Thus the tangent of the angle of polarisation is equal to the refractive index at reflecting medium. It is called Brewster's law.

Polarisation by refraction:

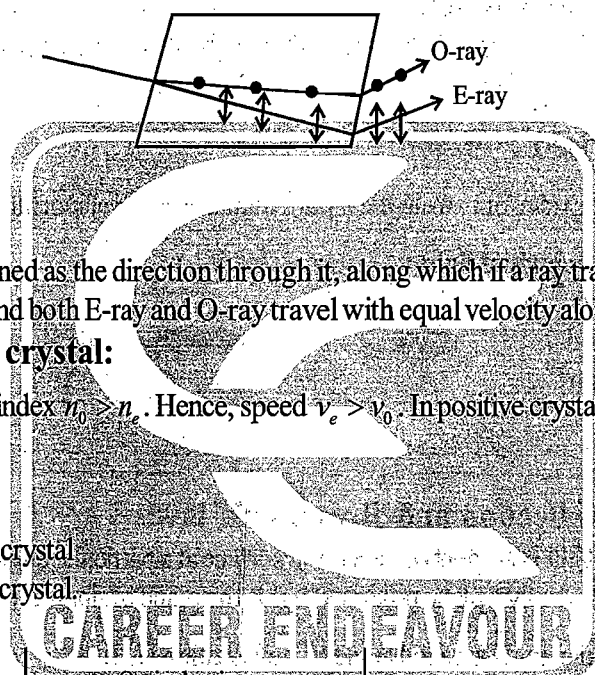
If a beam of light is incident on a glass plate at the polarising angle i_p , the transmitted light will be partially plane polarised. If we use number of parallel glass plates then at each reflection

as well as reduce amount of vibrations perpendicular to the plane of incidence. Thus, after large number of parallel plate the transmitted light become wholly polarised.



Polarisation by double refraction:

When an ordinary ray of unpolarised light is made incident on a crystal like calcite quartz etc. it is refracted into two rays. One of these rays obeys the laws of refraction and it is called ordinary ray. While the other ray does not and it is called extraordinary ray. Both O-ray and E-rays are plane polarised whose vibration are along and at right angles to the principle section. This phenomenon known as birefringence or double refraction.



Optical axis :

Optical axis of a crystal is defined as the direction through it, along which if a ray travels then there will be no double refraction of the ray and both E-ray and O-ray travel with equal velocity along this direction.

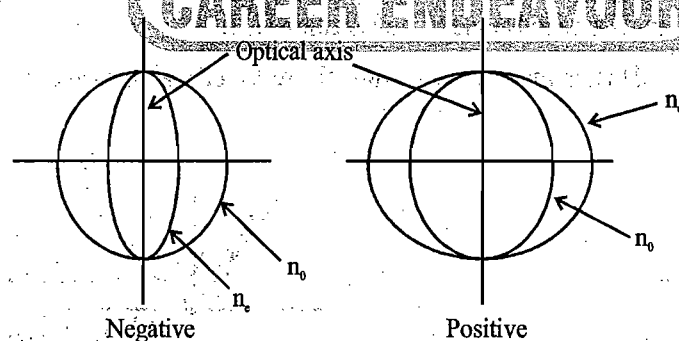
Negative and positive crystal:

In negative crystal refractive index $n_o > n_e$. Hence, speed $v_e > v_o$. In positive crystal refractive index $n_o < n_e$.

Hence, speed $v_o > v_e$

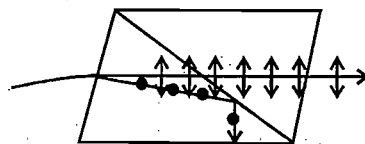
Example :

- (i) Calcite is negative uniaxial crystal.
- (ii) Quartz is positive uniaxial crystal.



Surface index of positive and negative crystal.

Polariser : Polariser is a crystal which produce plane polarised light.



Malus's law:

If θ be the angle between direction of electric vector and polarisation axis and I_0 be the intensity of incident light. Then the intensity of transmitted light is



Superposition of two plane polarised waves vibrating in mutually perpendicular plans :

Suppose a plane polarised monochromatic light be incident normally on a calcite plate. If the incident wave is represented by

$$E = A \cos(\omega t - kz),$$

then the component waves are

$$x = a \cos(\omega t - kz)$$

$$y = b \cos(\omega t - kz)$$

due to different velocity of E and O-ray a phase introduce between then say δ . In the emergent rays are

$$x = a \cos(\omega t - kz + \delta)$$

$$y = b \cos(\omega t - kz)$$

$$\text{Now, } \frac{x}{a} = \cos(\omega t - kz + \delta)$$

$$\Rightarrow \frac{x}{a} = \cos(\omega t - kz) \cos \delta - \sin(\omega t - kz) \sin \delta$$

$$\Rightarrow \frac{x}{a} = \frac{y}{b} \cos \delta - \sqrt{1 - \frac{y^2}{b^2}} \sin \delta$$

$$\Rightarrow \left(\frac{x}{a} - \frac{y}{b} \cos \delta \right)^2 = \left(1 - \frac{y^2}{b^2} \right) \sin^2 \delta$$

$$\Rightarrow \frac{x^2}{a^2} - \frac{2xy}{ab} \cos \delta + \frac{y^2}{b^2} = \sin^2 \delta \quad (65)$$

Special Cases:

(1) If $\delta = 2m\pi, m = 0, 1, 2, \dots$

$$y = \frac{b}{a} x = \tan \theta \cdot x \quad \dots (66)$$

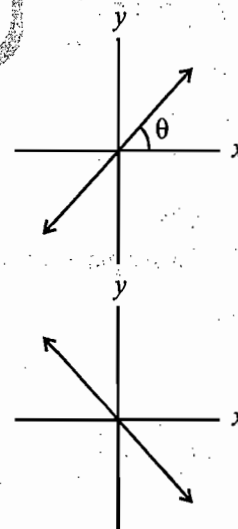
(2) If $\delta = (2m+1)\pi, m = 0, 1, 2,$

$$y = \frac{b}{a} x = -\tan \theta x \quad \dots (67)$$

(3) If $\delta = (2m+1)\frac{\pi}{2}, m = 0, 1, 2$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \dots (68)$$

Output will be elliptically polarised.





$$\left. \begin{aligned} x &= -a \sin \omega t \\ y &= b \cos \omega t \end{aligned} \right\} \text{left handed elliptical}$$

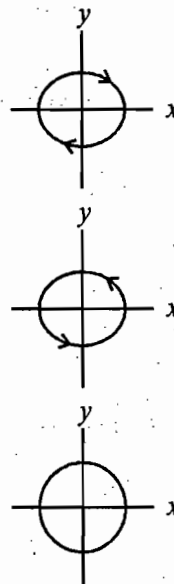
$$\text{If } \delta = \frac{3\pi}{2}, \frac{7\pi}{2}$$

$$\left. \begin{aligned} x &= a \sin \omega t \\ y &= b \cos \omega t \end{aligned} \right\} \text{right-handed elliptically}$$

$$(4) \text{ If } a = b, \delta = (2m+1)\frac{\pi}{2}$$

$$x^2 + y^2 = a^2 \quad \dots(69)$$

The emergent light is then said to be circularly polarised.



Retardation plates :

A plate made of doubly refracting crystal introduces phase between O-ray and E-ray
The path difference between O-ray and E-ray is

$$(n_o \sim n_e) d$$

Corresponding phase difference

$$\delta = \frac{2\pi}{\lambda} (n_o \sim n_e) d$$

(1) Quarter-wave plate

$$\text{For quarter wave plate, } \delta = \frac{\pi}{2}$$

$$\frac{2\pi}{\lambda} (n_o \sim n_e) d = \frac{\pi}{2}$$

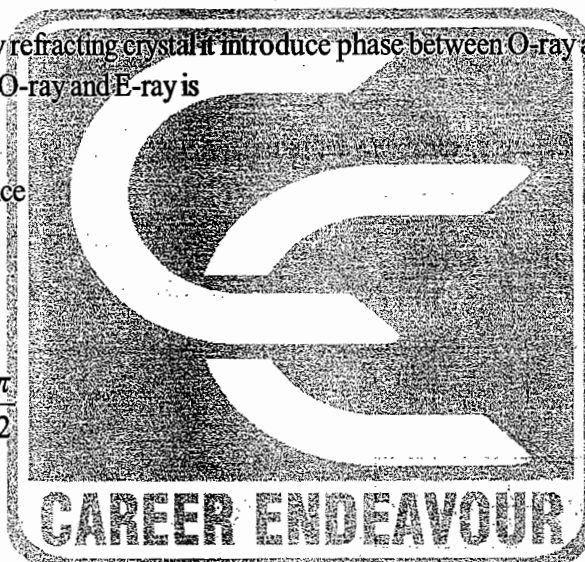
$$\Rightarrow \boxed{d = \frac{\lambda}{4(n_o \sim n_e)}} \quad \dots (70)$$

(2) Half-wave plate :

$$\text{For half-wave plate } \delta = \pi$$

$$\therefore \frac{2\pi}{\lambda} (n_o \sim n_e) d = \pi$$

$$\Rightarrow \boxed{d = \frac{\lambda}{2(n_o \sim n_e)}} \quad \dots (71)$$





SOLVED PROBLEMS

1. The electric field $E(\mathbf{r}, t)$ at a point $\mathbf{r}$ at time t in a metal due to the passage of electrons can be described by the equation [GATE 2001]

$$\nabla^2 \vec{E}(\mathbf{r}, t) = \frac{1}{c^2} \left[\frac{\partial^2 \vec{E}(\mathbf{r}, t)}{\partial t^2} + \omega'^2 \vec{E}(\mathbf{r}, t) \right]$$

Where ω' is a characteristic associated with the metal and c is the speed of light in vacuum. The dispersion relation corresponding to the plane wave solutions of the form $\exp[i(\mathbf{k} \cdot \mathbf{r} - \omega t)]$ is given by:

- (a) $\omega^2 = c^2 k^2 - \omega'^2$ (b) $\omega^2 = c^2 k^2 + \omega'^2$ (c) $\omega^2 = ck - \omega'$ (d) $\omega = ck + \omega'$

Soln. Let $E(\mathbf{r}, t) = E_0 e^{i(\mathbf{k} \cdot \mathbf{r} - \omega t)}$

We have,
$$\nabla^2 E(\mathbf{r}, t) = \frac{1}{c^2} \left[\frac{\partial^2 E(\mathbf{r}, t)}{\partial t^2} + \omega'^2 E(\mathbf{r}, t) \right]$$

Replacing ∇ by $i\mathbf{k}$ and $\frac{\partial}{\partial t} = -i\omega$ we get

$$-k^2 = \frac{1}{c^2} [-\omega^2 + \omega'^2] \quad (\text{since } E_0 e^{i(\mathbf{k} \cdot \mathbf{r} - \omega t)} \neq 0)$$

$$\Rightarrow k^2 c^2 = \omega^2 - \omega'^2$$

$$\Rightarrow \omega^2 = c^2 k^2 + \omega'^2$$

Correct option is (b)

2. An electromagnetic wave is propagating in free space in the z -direction. If the electric field is given by $E = \cos(\omega t - kz)\mathbf{j}$, where $\omega t = ck$, then the magnetic field is given by [GATE 2003]

- (a) $B = (1/c)\cos(\omega t - kz)\mathbf{j}$ (b) $B = (1/c)\sin(\omega t - kz)\mathbf{j}$
 (c) $B = (1/c)\cos(\omega t - kz)\mathbf{i}$ (d) $B = (1/c)\cos(\omega t - kz)\mathbf{j}$

Soln. We know the relation between $\vec{E}$, $\vec{k}$ and $\vec{B}$ is

$$\vec{k} \times \vec{E} = \omega \vec{B}$$

$$\Rightarrow \vec{B} = \frac{1}{\omega} (\vec{k} \times \vec{E}) = \frac{k}{\omega} (\vec{k} \times \vec{E}) = \frac{1}{c} (\vec{k} \times \vec{E})$$

We have, $E = \cos(\omega t - kt)\mathbf{i}$

$$\therefore \hat{k} = \hat{z}$$

$$\therefore \vec{B} = \frac{1}{c} [\hat{z} \times \hat{i} \cos(\omega t - kt)] = \frac{1}{c} \cos(\omega t - kt) \hat{j}$$

Correct option is (a)

3. Which one of the following Maxwell's equations implies the absence of magnetic monopoles?

(a) $\vec{\nabla} \cdot \vec{E} = \rho / \epsilon_0$

(b) $\vec{\nabla} \cdot \vec{B} = 0$

[GATE 2003]



Soln. $\vec{\nabla} \cdot \vec{B} = 0$

$$\Rightarrow \int \vec{\nabla} \cdot \vec{B} dv = 0$$

$$\Rightarrow \oint \vec{B} \cdot d\vec{S} = 0$$

$$\Rightarrow \phi = 0$$

That say that monopole does not exist.

Correct option is (b)

4. In a two beam interference pattern, the maximum and minimum intensity values are found to be $25 I_0$ and $9 I_0$ respectively, where I_0 is a constant. The intensities of the two interfering beams are : [GATE 2002]

- (a) $16 I_0$ and I_0 (b) $5 I_0$ and $3 I_0$ (c) $17 I_0$ and $8 I_0$ (d) $8 I_0$ and $2 I_0$

Soln. $\frac{I_{\max}}{I_{\min}} = \frac{25 I_0}{9 I_0} = \frac{25}{9}$

$$\left(\frac{a+b}{a-b} \right)^2 = \frac{25}{9} \Rightarrow \frac{a+b}{a-b} = \pm \frac{5}{3}$$

Taking positive value

$$3a + 3b = 5a - 5b$$

$$2a = 8b \Rightarrow \frac{a}{b} = \frac{4}{1}$$

$$\therefore \frac{I_1}{I_2} = \left(\frac{a}{b} \right)^2 = \frac{16 I_0}{I_0}$$

Correct option is (a)

5. A left circularly polarized light beam of wavelength 600 nm is incident on a crystal of thickness d and propagated perpendicular to its optic axis. The ordinary and extraordinary refractive indices of the crystal are $n_o = 1.54$ and $n_e = 1.55$ respectively. The emergent light will be right circularly polarized if d is :

[GATE 2002]

- (a) $120 \mu\text{m}$ (b) $60 \mu\text{m}$ (c) $30 \mu\text{m}$ (d) $15 \mu\text{m}$

Soln. Two components of left circular light as

$$E_x = E_0 \cos(\omega t - kz)$$

$$E_y = E_0 \sin(\omega t - kz)$$

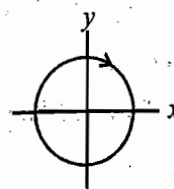
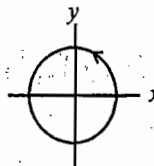
Let after travelling 'd' distance the left circular polarized light become right circular polarized.

$$\therefore E_x = E_0 \cos(\omega t - kz)$$

$$E_y = -E_0 \sin(\omega t - kz)$$

Therefore, the phase introduce between two component

$$\delta = \frac{2\pi}{\lambda} (n_e - n_o) d = \pi$$





$$\Rightarrow d = \frac{\lambda}{2(n_e - n_o)} = \frac{600}{2 \times 0.01} = 30 \mu m$$

Correct option is (c)

Common Data for Q. 6 and Q. 7

Let $\vec{E} = \hat{x}E_0 \exp[i\vec{k} \cdot \vec{r} - \omega t]$, where $\vec{k} = \hat{z}(k \cos \phi + ik \sin \phi)$, $k = 1\bar{k}$ and $\hat{x}$, $\hat{y}$ and $\hat{z}$ are cartesian unit vectors, represent an electric field of a plane electro magnetic wave of frequency ω .

Which one of the following statements is TRUE?

[GATE 2004]

- (a) The magnitude of the electric field is attenuated as the wave propagates
- (b) The energy of the e.m. wave flows along the x-direction
- (c) The magnitude of the electric field of the wave is a constant
- (d) The speed of the wave is the same as c. (speed of light in free space)

$$\vec{E} = \hat{x}E_0 e^{i(\vec{k} \cdot \vec{r} - \omega t)}$$

$$\vec{k} = \hat{z}(k \cos \phi + ik \sin \phi)$$

$$E = \hat{x}E_0 e^{i[\hat{z}(k \cos \phi + ik \sin \phi) \cdot \vec{r} - \omega t]}$$

$$= \hat{x}E_0 e^{-k \sin \phi z} e^{i(k \cos \phi z - \omega t)}$$

This says that the amplitude decay as the propagation.

Correct option is (a)

The magnetic field $\vec{B}$ of the wave is:

[GATE 2004]

$$(a) \hat{y} \frac{k}{\omega} E_0 \exp(-zk \sin \phi) \exp[i(zk \cos \phi - \omega t)]$$

$$(b) \hat{y} \frac{k}{\omega} E_0 \exp(-zk \sin \phi) \exp[i(zk \cos \phi - \omega t + \phi)]$$

$$(c) \hat{y} \frac{k}{\omega} E_0 \exp[i(zk \cos \phi - \omega t + \phi)]$$

$$(d) \hat{y} \frac{k}{\omega} E_0 \exp(-zk \cos \phi) \exp[i(zk \sin \phi - \omega t)]$$

Soln. We know,

$$\vec{k} \times \vec{E} = \omega \vec{B}$$

$$\Rightarrow \vec{B} = \frac{k}{\omega} (\hat{k} \times \vec{E})$$

We have,

$$\vec{E} = \hat{x}E_0 e^{i(\vec{k} \cdot \vec{r} - \omega t)} = \hat{x}E_0 e^{-k \sin \phi z} e^{i(k \cos \phi z - \omega t)} \text{ and } \hat{k} = \hat{z}$$



$$\begin{aligned}\vec{B} &= \frac{k}{\omega} E_0 e^{-kt \sin \varphi} e^{i(kt \cos \varphi - \omega t)} [\hat{z} \times \hat{x}] \\ &= \hat{y} \frac{k}{\omega} E_0 e^{-kt \sin \varphi} e^{i(kt \cos \varphi - \omega t)}\end{aligned}$$

Correct option is (b)

8. The state of polarization of light with the electric field vector

[GATE 2004]

$$\vec{E} = \hat{i} E_0 \cos(kz - \omega t) - \hat{j} E_0 \cos(kz - \omega t) \text{ is :}$$

- (a) Linearly polarized along z-direction
- (b) Linearly polarized at -45° to x-axis
- (c) Circularly polarized
- (d) Elliptically polarized with the major axis along x-axis

Soln. We have,

$$\vec{E} = \hat{i} E_0 \cos(kz - \omega t) - \hat{j} E_0 \cos(kz - \omega t)$$

So, $E_x = E_0 \cos(kz - \omega t)$

$$E_y = -E_0 \cos(kz - \omega t)$$

$$\therefore \frac{E_y}{E_x} = -1$$

$$\therefore E_y = -E_x$$

Therefore, it is a linear polarized light at -45° to x-axis.

Data for Q. 9 and Q. 10.

Consider two conducting plates of infinite extent, one plate at $z = 0$ and the other at $z = L$, both parallel to the xy plane. The vector and scalar potential in the region between the plates is given by :

$$A(r, t) = A_0 \hat{i} \cos(kz + \alpha) \cos(kct), \quad \phi(r, t) = 0$$

9. For this to represent a standing wave in the empty region between the plates :

[GATE 2003]

- (a) $k = \pi/L$ and $\alpha = 0$
- (b) $k = 2\pi/L$ and $\alpha = \pi/2$
- (c) $k = \pi/(2L)$ and $\alpha = \pi/2$
- (d) $k = \pi/2L$ and $\alpha = 0$

Soln. We have,

$$A(r, t) = A_0 \hat{i} \cos(kz + \alpha) \cos(kct), \quad \phi(r, t) = 0$$

$$\therefore \vec{E} = -\frac{\partial \vec{A}}{\partial t} - \nabla \phi = -\frac{\partial A}{\partial t} + 0 = +A_0 kc \cos(kz + \alpha) \sin(kct)$$

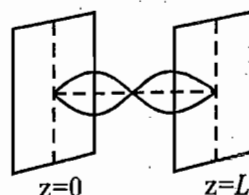
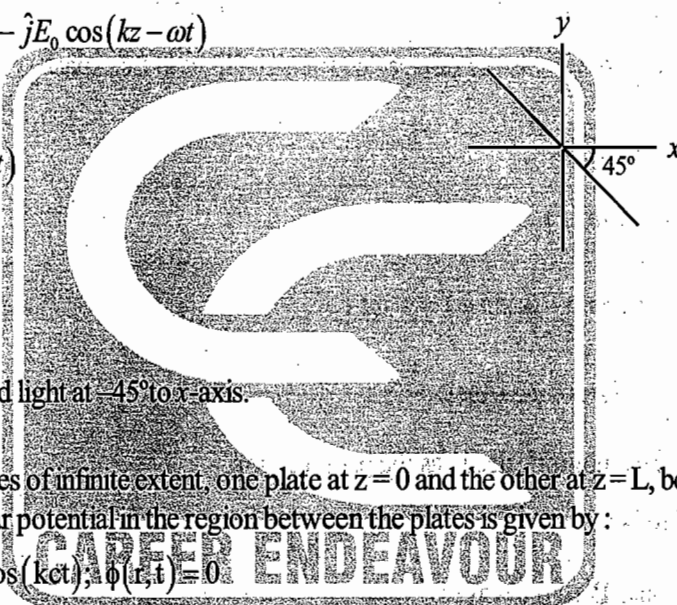
For standing wave the boundary condition is

$$E(0, t) = E(L, t) = 0$$

$$\therefore \cos \alpha = 0; \quad \alpha = \frac{\pi}{2}$$

And $\cos(kL + \alpha) = 0$

$$(kL + \alpha) = (2m + 1) \times \frac{\pi}{2} \quad (m = 0, 1, 2, 3, \dots)$$





$$kL + \frac{\pi}{2} = 2\pi + \frac{\pi}{2} = \frac{5\pi}{2} \Rightarrow k = \left(\frac{2\pi}{L}\right)$$

$$\therefore k = \frac{2\pi}{L} \text{ and } \alpha = \frac{\pi}{2}$$

Correct option is (b)

10. The energy density at $z=0$ and $t=0$ is :

[GATE 2003]

- (a) 0 (b) $\epsilon_0 c^2 k^2 A_0^2$
(c) $(1/2)\mu_0 A_0^2 k^2$ (d) $(1/2)\mu_0 A_0^2 k^2 + (1/2)\epsilon_0 c^2 k^2 A_0^2$

Soln.

11. In a non-conducting medium characterized by $\epsilon = \epsilon_0$, $\mu = \mu_0$ and conductivity $\sigma = 0$, the electric field (in Vm^{-1}) is given by $\vec{E} = 20 \sin[10^8 t - kz] \hat{j}$. The magnetic field, $\vec{H}$ (in A m^{-1}), is given by:

- (a) $20k \cos[10^8 t - kz] \hat{j}$ (b) $\frac{20k}{10^8 \mu_0} \sin[10^8 t - kz] \hat{j}$ [GATE 2009]
(c) $-\frac{20k}{10^8 \mu_0} \sin[10^8 t - kz] \hat{i}$ (d) $-20k \cos[10^8 t - kz] \hat{j}$

Soln. We know the relation between $\vec{E}$, $\vec{k}$, $\vec{H}$ is

$$\vec{k} \times \vec{E} = \omega \mu_0 \vec{H}$$

$$\vec{H} = \frac{k}{\omega \mu_0} (\vec{k} \times \vec{E})$$

We have,

$$\vec{E} = 20 \sin(10^8 t - kz) \hat{j}$$

$$\hat{k} = \hat{z}, \omega = 10^8$$

$$\therefore \vec{H} = \frac{20k}{\omega \mu_0} (\hat{z} \times \hat{j}) \sin(10^8 t - kz) = -\frac{20k}{10^8 \mu_0} \sin(10^8 t - kz) \hat{i}$$

Correct option is (c)

12. Which one of the following sets of Maxwell's equations for time - independent charge density ρ and current density $\hat{J}$ is correct?

[GATE 2006]

- (a) $\vec{\nabla} \cdot \vec{E} = \rho / \epsilon_0$ (b) $\vec{\nabla} \cdot \vec{E} = \rho / \epsilon_0$ (c) $\vec{\nabla} \cdot \vec{E} = 0$ (d) $\vec{\nabla} \cdot \vec{E} = \rho / \epsilon_0$
 $\vec{\nabla} \cdot \vec{B} = 0$ $\vec{\nabla} \cdot \vec{B} = 0$ $\vec{\nabla} \cdot \vec{B} = 0$ $\vec{\nabla} \cdot \vec{B} = \mu_0 \hat{J}$
 $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$ $\vec{\nabla} \times \vec{E} = 0$ $\vec{\nabla} \times \vec{E} = 0$ $\vec{\nabla} \times \vec{E} = 0$
 $\vec{\nabla} \times \vec{B} = \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$ $\vec{\nabla} \times \vec{B} = \mu_0 \hat{J}$ $\vec{\nabla} \times \vec{B} = \mu_0 \hat{J}$ $\vec{\nabla} \times \vec{B} = \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$

Soln. The Maxwell's equations are

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}$$



$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \epsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t} = \mu_0 \vec{J} + \mu_0 \frac{\partial \vec{B}}{\partial t}$$

For steady state circuit flow $\frac{\partial B}{\partial t} = 0$

$$\therefore \nabla \times B = \mu_0 J \text{ and since } J \text{ is time independent } \therefore \frac{\partial B}{\partial t} = 0$$

Therefore, the Maxwell's equation for time independent charge density and current density are

$$\vec{\nabla} \cdot \vec{E} = \rho / \epsilon_0$$

$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{E} = 0$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$$

Correct option is (b)

13. A parallel plate capacitor is being discharged. What is the direction of the energy flow in terms of the poynting vector in the space between the plates? [GATE 2008]



- (a) Along the wire in the positive z axis (b) Radially inward ($-\hat{r}$)
(c) Radially outward ($\hat{r}$) (d) Circumferential ($\hat{\phi}$)

Soln. The electric field inside the two parallel plate is along $-\hat{z}$ and from right handed rule we can say the magnetic field along $\hat{\phi}$.

Therefore, poynting vector, $\vec{s} = \vec{E} \times \vec{H}^*$

Therefore, direction of poynting $\vec{s} = -\hat{z} \times \hat{\phi}$

Velocity = $\hat{r}$

Correct option is (c)

14. Unpolarized light falls from air to a planar air-glass interface (refractive index of glass is 1.5) and the reflected light is observed to be plane polarized. The polarization vector and the angle of incidence q are: [GATE 2008]

- (a) Perpendicular to the plane of incidence and $\theta_i = 42^\circ$
(b) Parallel to the plane of incidence and $\theta_i = 56^\circ$
(c) Perpendicular to the plane of incidence and $\theta_i = 56^\circ$

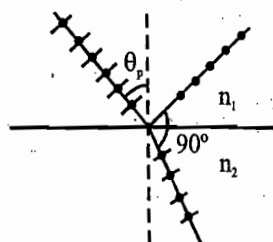


Soln. We know that at $\theta = \theta_p$

$$\tan \theta_p = \frac{n_2}{n_1} = \frac{1.5}{1}$$

$$\Rightarrow \theta_p = \tan^{-1}(1.5) = 50^\circ$$

Correct option is (c)



Linked Answer Questions:

Statement for Linked Answer Q.15 and Q.16.

Consider the propagation of electromagnetic waves in a linear, homogenous and isotropic material medium with electric permittivity ϵ , and magnetic permeability μ .

15. For a plane wave of angular frequency ω and propagation vector $\vec{k}$ propagating in the medium Maxwell's equations reduce to [GATE 2010]

- (a) $\vec{k} \cdot \vec{E} = 0; \vec{k} \cdot \vec{H} = 0; \vec{k} \times \vec{E} = \omega \epsilon \vec{H}; \vec{k} \times \vec{H} = -\omega \mu \vec{E}$
 (b) $\vec{k} \cdot \vec{E} = 0; \vec{k} \cdot \vec{H} = 0; \vec{k} \times \vec{E} = -\omega \epsilon \vec{H}; \vec{k} \times \vec{H} = \omega \mu \vec{E}$
 (c) $\vec{k} \cdot \vec{E} = 0; \vec{k} \cdot \vec{H} = 0; \vec{k} \times \vec{E} = -\omega \mu \vec{H}; \vec{k} \times \vec{H} = \omega \epsilon \vec{E}$
 (d) $\vec{k} \cdot \vec{E} = 0; \vec{k} \cdot \vec{H} = 0; \vec{k} \times \vec{E} = \omega \mu \vec{H}; \vec{k} \times \vec{H} = -\omega \epsilon \vec{E}$

Soln. Let $\vec{E} = \vec{E}_0 e^{i(\vec{k} \cdot \vec{r} - \omega t)}$

We know that Maxwell's equations,

$$\vec{\nabla} \cdot \vec{E} = 0$$

$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla} \times \vec{B} = \epsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t}$$

Replacing $\vec{\nabla}$ by $i\vec{k}$ and $\frac{\partial}{\partial t}$ by $-i\omega$, we get

$$i\vec{k} \cdot \vec{E} = 0 \Rightarrow \vec{k} \cdot \vec{E} = 0$$

$$i\vec{k} \cdot \vec{B} = 0 \Rightarrow \vec{k} \cdot \vec{B} = 0$$

$$i\vec{k} \times \vec{E} = i\omega \mu \vec{H} \Rightarrow \vec{k} \times \vec{E} = \omega \mu \vec{H}$$

$$i\vec{k} \times \vec{H} = i\omega \epsilon \vec{E} \Rightarrow \vec{k} \times \vec{H} = -\omega \epsilon \vec{E}$$

Correct option is (d)

16. If ϵ and μ assume negative values in a certain frequency range, then the directions of the propagation vector $\vec{k}$ and the Poynting vector $\vec{S}$ in that frequency range are related as

- (a) $\vec{k}$ and $\vec{S}$ are parallel
 (b) $\vec{k}$ and $\vec{S}$ are anti-parallel
 (c) $\vec{k}$ and $\vec{S}$ are perpendicular to each other

[GATE 2010]



Soln. $\vec{S} = \vec{E} \times \vec{H}^* = \vec{E} \times (\vec{k} \times \vec{E}^*) \frac{1}{\mu\omega}$

$$= \frac{1}{\mu\omega} [\vec{k} (\vec{E} \cdot \vec{E}^*) - (\vec{k} \cdot \vec{E}) \vec{E}^*] = \frac{1}{\mu\omega} \vec{k} |E|^2 = \frac{k}{\mu\omega} |E|^2 \hat{k}$$

Since μ is negative.

$$\vec{S} = -\frac{k}{\mu\omega} |E|^2 \hat{k}$$

Therefore, k and S are anti-parallel.

Correct option is (b)

Statement for Linked Answer Q. 17 and Q. 18:

A plane electromagnetic wave has the magnetic field given by $\vec{B}(x, y, z, t) = B_0 \sin \left[(x+y) \frac{k}{\sqrt{2}} + \omega t \right] \hat{k}$

Where k is the wave number and $\hat{i}$, $\hat{j}$ and $\hat{k}$ are the Cartesian unit vectors in x , y and z directions, respectively.

17. The electric field $\vec{E}(x, y, z, t)$ corresponding to the above wave is given by [GATE 2011]

(a) $cB_0 \sin \left[(x+y) \frac{k}{\sqrt{2}} + \omega t \right] \frac{(\hat{j} - \hat{i})}{\sqrt{2}}$ (b) $cB_0 \sin \left[(x+y) \frac{k}{\sqrt{2}} + \omega t \right] \frac{(\hat{i} + \hat{j})}{\sqrt{2}}$

(c) $cB_0 \sin \left[(x+y) \frac{k}{\sqrt{2}} + \omega t \right] \hat{i}$ (d) $cB_0 \sin \left[(x+y) \frac{k}{\sqrt{2}} + \omega t \right] \hat{j}$

Soln. $\vec{B} = B_0 \sin \left[(x+y) \frac{k}{\sqrt{2}} + \omega t \right] \hat{k}$

We know the solution,

$$\begin{aligned} \vec{E} &= -\frac{1}{\omega\epsilon_0\mu_0} [\vec{k} \times \vec{B}] = -\frac{k}{\omega} c^2 [\hat{k} \times \vec{B}] \\ &= -cB_0 \sin \left[(x+y) \frac{k}{\sqrt{2}} + \omega t \right] \left[(\hat{i} + \hat{j}) \frac{1}{\sqrt{2}} \times \hat{k} \right] \\ &= cB_0 \sin \left[(x+y) \frac{k}{\sqrt{2}} + \omega t \right] (\hat{j} - \hat{i}) \frac{1}{\sqrt{2}} \\ &= cB_0 \frac{(\hat{j} - \hat{i})}{\sqrt{2}} \sin \left[(x+y) \frac{k}{\sqrt{2}} + \omega t \right] \end{aligned}$$

Correct option is (a)

18. The average Poynting vector is given by [GATE 2011]

(a) $\frac{cB_0^2}{2\mu_0} \frac{(\hat{i} - \hat{j})}{\sqrt{2}}$ (b) $\frac{cB_0^2}{2\mu_0} \frac{(\hat{i} - \hat{j})}{\sqrt{2}}$ (c) $\frac{cB_0^2}{2\mu_0} \frac{(\hat{i} + \hat{j})}{\sqrt{2}}$ (d) $\frac{cB_0^2}{2\mu_0} \frac{(\hat{i} + \hat{j})}{\sqrt{2}}$



Soln. $\langle S \rangle = \langle (E \times H^*) \rangle = \frac{cB_0^2}{T\mu_0} \int \sin^2 \left[(x+y) \frac{k}{\sqrt{2}} + \omega t \right] \frac{(\hat{i} + \hat{j})}{\sqrt{2}} dt$

$$= \frac{cB_0^2}{2\sqrt{2}\mu_0} (\hat{i} + \hat{j})$$

Correct option is (c)

19. The space-time dependence of the electric field of a linearly polarized light in free space is given by $\hat{x}E_0 \cos(\omega t - kz)$ where E_0 , ω and k are the amplitude, the angular frequency and the wavevector, respectively. The time averaged energy density associated with the electric field is [GATE 2012]

- (a) $\frac{1}{4} \epsilon_0 E_0^2$ (b) $\frac{1}{2} \epsilon_0 E_0^2$ (c) $\epsilon_0 E_0^2$ (d) $2\epsilon_0 E_0^2$

Soln. We have,

$$\vec{E} = E_0 \cos(\omega t - kt) \hat{x}$$

Therefore, the time average energy density associated with electric field

$$\langle u \rangle = \frac{1}{T} \int E^2 dt = \frac{1}{T} \int \frac{\epsilon_0 E_0^2}{2} \cos^2(\omega t - kt) dt = \frac{1}{2} \epsilon_0 E_0^2 \frac{1}{2} = \frac{1}{4} \epsilon_0 E_0^2$$

Correct option is (a)

20. Interference fringes are seen at an observation plane $z=0$, by the superposition of two plane waves $A_1 \exp[i(\vec{k}_1 \cdot \vec{r} - \omega t)]$ and $A_2 \exp[i(\vec{k}_2 \cdot \vec{r} - \omega t)]$, where A_1 and A_2 are real amplitudes. The condition for interference maximum is [GATE 2013]

- (a) $(\vec{k}_1 - \vec{k}_2) \cdot \vec{r} = (2m+1)\pi$ (b) $(\vec{k}_1 - \vec{k}_2) \cdot \vec{r} = 2m\pi$
 (c) $(\vec{k}_1 + \vec{k}_2) \cdot \vec{r} = (2m+1)\pi$ (d) $(\vec{k}_1 + \vec{k}_2) \cdot \vec{r} = 2m\pi$

Soln. The phase difference between two wave is

$$\phi = (\vec{k}_1 \cdot \vec{r} - \vec{k}_2 \cdot \vec{r}) = (\vec{k}_1 - \vec{k}_2) \cdot \vec{r}$$

We know that the condition for maximum is

$$\phi = 2m\pi, \quad \text{where } m = 0, 1, 2, \dots$$

$$\therefore (\vec{k}_1 - \vec{k}_2) \cdot \vec{r} = 2m\pi$$

Correct option is (b)

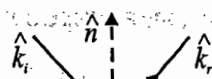
21. A monochromatic plane wave at oblique incidence undergoes reflection at a dielectric interface. If $\hat{k}_i$, $\hat{k}_r$ and $\hat{n}$ are the unit vectors in the directions of incident wave, reflected wave and the normal to the surface respectively, which one of the following expression is correct? [GATE 2013]

- (a) $(\hat{k}_i - \hat{k}_r) \times \hat{n} \neq 0$ (b) $(\hat{k}_i - \hat{k}_r) \cdot \hat{n} = 0$ (c) $(\hat{k}_i \times \hat{n}) \cdot \hat{k}_r = 0$ (d) $(\hat{k}_i \times \hat{n}) \cdot \hat{k}_r \neq 0$

Soln. We know that $\vec{k}_i, \vec{k}_r, \hat{n}$ all are in the same plane.

The vector $(\hat{k}_i \times \hat{n})$ have direction perpendicular to the plane,

that contain by the $\hat{k}_i, \hat{n}, \hat{k}_r$





$$\therefore (\hat{k}_i \times \hat{n}) \cdot \hat{k}_r = 0 \text{ because } (\hat{k}_i \times \hat{n}) \perp \hat{k}_r$$

Correct option is (c)

22. The electric field of a uniform plane wave propagating in a dielectric, non-conducting medium is given by,

$$\vec{E} = \hat{x} 10 \cos(6\pi \times 10^7 t - 0.4\pi z) \text{ V/m}$$

The phase velocity of the wave is _____ $\times 10^8$ m/s.

[GATE 2014]

Soln. We have, $\vec{E} = \hat{x} 10 \cos(6\pi \times 10^7 t - 0.4\pi z) \frac{\text{V}}{\text{m}}$

Compared with electric field $\vec{E} = \hat{x}(\omega t - kz) \frac{\text{V}}{\text{m}}$

We get, $\omega = 6\pi \times 10^7$, $k = 0.4\pi$

Phase velocity, $V_p = \frac{\omega}{k} = \frac{6\pi \times 10^7}{0.4\pi} = 1.5 \times 10^8 \text{ m/sec}$

23. In an interference pattern formed by two coherent sources, the maximum and the minimum of the intensities are $9I_0$ and I_0 , respectively. The intensities of the individual waves are [GATE 2014]

- (a) $3I_0$ and I_0 (b) $4I_0$ and I_0 (c) $5I_0$ and $4I_0$ (d) $9I_0$ and I_0

Soln. $\frac{I_{\max}}{I_{\min}} = \left(\frac{a+b}{a-b}\right)^2 = \frac{9I_0}{I_0} = 9 \Rightarrow \frac{a+b}{a-b} = \pm 3$

Taking positive value, we get

$$a+b = 3a-3b$$

$$\Rightarrow 2a = 4b \Rightarrow \frac{a}{b} = \frac{2}{1}$$

$$\therefore \frac{I_1}{I_2} = \left(\frac{a}{b}\right)^2 = \frac{4}{1} = \frac{4I_0}{I_0}$$

Correct option is (b)

24. An electromagnetic wave with $\vec{E}(z, t) = E_0 \cos(\omega t - kz) \hat{i}$ is travelling in free space and crosses a disc of radius 2m placed perpendicular to the z-axis. If $E_0 = 90 \text{ V m}^{-1}$, the average power, in watt, crossing the disc along the z-direction is [GATE 2007]

- (a) 30 (b) 60 (c) 120 (d) 270

Soln. We have, $I = 150 \frac{\text{W}}{\text{m}^2}$

We know, $I = \langle u \rangle c = \frac{1}{2} \epsilon_0 E_0^2 c$

Therefore, the average power $= I \times \pi R^2 = \frac{1}{2} \times 8.85 \times 10^{-12} \times (90)^2 \times \pi \times (2)^2 \times 3 \times 10^8 = 135 \times 120$

Correct option is (c)

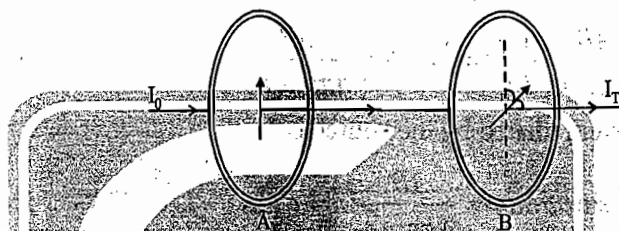


25. The intensity of a laser in free space is 150 mW/m^2 . The corresponding amplitude of the electric field of the laser is _____ V/m. ($\epsilon_0 = 8.854 \times 10^{-12} \text{ C}^2/\text{N.m}^2$) [GATE 2014]

Soln. We know that, $I = \langle u \rangle c = \frac{1}{2} \epsilon_0 E_0^2 c$

$$E_0 = \sqrt{\frac{2I}{c\epsilon_0}} = \sqrt{\frac{2 \times 150 \times 10^{-3}}{3 \times 10^8 \times 8.85 \times 10^{-12}}} = 10.62 \frac{\text{V}}{\text{m}}$$

26. Unpolarised light of intensity I_0 Passes successively through two identical linear polarisers A and B, placed such that their polarisation axes are at an angle of 45° (see figure) with respect to one another. [TIFR 2012]



Assuming A and B to be perfect polarisers (i.e. no absorption losses), the intensity of the transmitted light will be $I_T =$

(a) $I_0/4$

(b) $I_0/2\sqrt{2}$

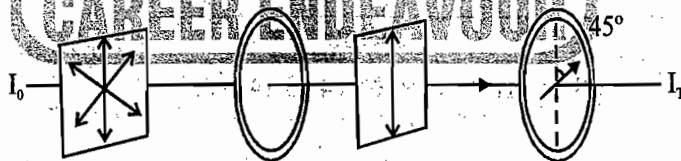
(c) $I_0/2$

(d) $I_0/\sqrt{2}$

Soln. The intensity after passing through the polariser A is

$$I = \frac{I_0}{2}$$

Therefore, according to Malus's law,



The intensity of output is

$$I_T = \frac{I_0}{2} \cos^2 \theta = \frac{I_0}{2} \cos^2 45^\circ = \frac{I_0}{2} \left(\frac{1}{\sqrt{2}} \right)^2 = \frac{1}{4} I_0$$

Correct option is (a)

27. A plane electromagnetic wave travelling in a vacuum is characterised by the electric and magnetic fields

$$\vec{E} = \hat{i} (30\pi \text{ Vm}^{-1}) \exp i(\omega t + kz)$$

$$\vec{H} = \hat{j} (H_0 \text{ Am}^{-1}) \exp i(\omega t + kz)$$

If $\omega, k > 0$, the value of H_0 must be

(a) 2 π

(b) 0.67

(c) 0.25

(d) 0.94

[TIFR 2013]

Soln. $|\vec{H}| = H_0 = \frac{|\vec{k} \times \vec{E}|}{\omega \mu_0} = \frac{kE_0}{\omega \mu_0} = \epsilon_0 c E_0 = 8.85 \times 10^{-12} \times 3 \times 10^8 \times 30\pi = 0.25$



28. The instantaneous electric and magnetic fields created at a distance 'r' by a point source at the origin are given by

$$\vec{E} = \frac{A \cos \omega t}{2\pi\epsilon_0 r} \hat{\theta} \quad \vec{H} = \frac{B \cos \omega t}{\mu_0 r} \hat{\phi}$$

[TIFR 2014]

where ω, A, B are constants, and the unit vectors $(\hat{r}, \hat{\theta}, \hat{\phi})$ form an orthonormal set. The time-averaged power radiated by the source is

- (a) $\frac{\omega\epsilon_0}{\mu_0} AB$ (b) $\frac{c^3}{2\pi} AB$ (c) $c^2 AB$ (d) $\frac{2\pi\omega}{c} AB$

Soln. We have,

$$\vec{E} = \frac{A \cos \omega t}{2\pi\epsilon_0 r} \hat{\theta} \quad \vec{H} = \frac{B \cos \omega t}{\mu_0 r} \hat{\phi}$$

$$\therefore \langle S \rangle = \langle \vec{E} \times \vec{H} \rangle = \frac{AB}{2\pi\epsilon_0 r^2 \mu_0} \frac{1}{T} \int_0^T \cos^2 \omega t dt = \frac{AB}{4\pi\epsilon_0 \mu_0 r^2}$$

Therefore, time-average power radiated by the source is

$$P = \int_0^\pi \int_0^{2\pi} \frac{AB}{4\pi\epsilon_0 \mu_0 r^2} r^2 \sin \theta d\theta d\phi = \frac{AB}{4\pi\epsilon_0 \mu_0} 4\pi = c^2 AB$$

Correct option is (c)

29. A light source has a small filament at the centre of a spherical glass bulb of radius 5 cm and negligible thickness. If this source emits 100 Watts of power in the form of spherical electromagnetic waves, the r.m.s. electric field E at the surface of the bulb (in units of Volt/m) will be approximately [TIFR 2015]
- (a) 1547 (b) 1094 (c) 109.4 (d) 15.47

Soln. We know power,

$$P = I \times 4\pi R^2 = \frac{1}{2} \epsilon_0 E_0^2 c \times 4\pi R^2$$

$$\Rightarrow E_0 = \sqrt{\frac{2P}{\epsilon_0 c \times 4\pi R^2}} = \sqrt{\frac{2 \times 100}{8.85 \times 10^{-12} \times 3 \times 10^8 \times 4 \times 3.14 \times (0.5)^2}} = 1094 \text{ volt/m}$$

Correct option is (b)

30. Measurement of the magnitudes of the electric field (E) and the magnetic field (B) in a plane polarized electromagnetic wave in vacuum leads to the following results [TIFR 2015]

$$\frac{\partial E}{\partial y} = -\frac{\partial B}{\partial t} \quad \frac{\partial B}{\partial y} = -\frac{1}{c^2} \frac{\partial E}{\partial t}$$

at all points where the measurement is made. In this case the electric vector $\vec{E}$, the magnetic vector $\vec{B}$ and the wave vector $\vec{k}$ (with magnetic k) can be written in terms of the unit vectors $(\hat{x}, \hat{y}, \hat{z})$ along the Cartesian axes as

- (a) $\vec{E} = E\hat{x}, \vec{B} = B\hat{y}, \vec{k} = k\hat{z}$ (b) $\vec{E} = E\hat{x}, \vec{B} = B\hat{z}, \vec{k} = -k\hat{y}$
 (c) $\vec{E} = E\hat{x}, \vec{B} = -B\hat{z}, \vec{k} = k\hat{y}$ (d) $\vec{E} = -E\hat{y}, \vec{B} = -B\hat{z}, \vec{k} = -k\hat{x}$

Soln. From the given two equations.

$$\left. \begin{aligned} \text{We can write, } \vec{k} \times \vec{E} &= \omega \vec{B} \\ \vec{k} \times \vec{B} &= -\left(\frac{1}{c^2}\right) \omega \vec{E} \end{aligned} \right\}$$



31. A time-dependent magnetic field $\vec{B}(t)$ is produced in a circular region of space, infinitely long and of radius R . The magnetic field is given as $\vec{B} = B_0 t \hat{z}$ for $0 \leq r < R$ and is zero for $r > R$. Where B_0 is a positive constant. The electric field for $r > R$ is [JEST 2012]

(a) $\frac{B_0 R^2}{r} \hat{r}$ (b) $\frac{B_0 R^2}{2r} \hat{\theta}$ (c) $-\frac{B_0 R^2}{r} \hat{r}$ (d) $-\frac{B_0 R^2}{2r} \hat{\theta}$

Soln. We have,

$$\vec{B} = B_0 t \hat{z} \quad \text{for } 0 < r < R$$

$$= 0 \quad r > R$$

Therefore, the electric field for $r > R$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} = 0 = \vec{\nabla} \times \left(-\frac{B_0 R^2}{2r} \hat{\theta} \right)$$

$$\therefore \vec{E} = -\frac{B_0 R^2}{2r} \hat{\theta}$$

Correct option is (d)

32. An electromagnetic wave of frequency ω travels in the x -direction through vacuum. It is polarized in the y -direction and the amplitude of the electric field is E_0 . With $k = \omega/c$ where c is the speed of light in vacuum, the electric and the magnetic fields are then conventionally given by [JEST 2013]

(a) $\vec{E} = E_0 \cos(ky - \omega t) \hat{x}$ and $\vec{B} = \frac{E_0}{c} \cos(ky - \omega t) \hat{z}$

(b) $\vec{E} = E_0 \cos(kx - \omega t) \hat{y}$ and $\vec{B} = \frac{E_0}{c} \cos(kx - \omega t) \hat{z}$

(c) $\vec{E} = E_0 \cos(kx - \omega t) \hat{z}$ and $\vec{B} = \frac{E_0}{c} \cos(kx - \omega t) \hat{y}$

(d) $\vec{E} = E_0 \cos(kx - \omega t) \hat{x}$ and $\vec{B} = \frac{E_0}{c} \cos(ky - \omega t) \hat{y}$

Soln. An electromagnetic wave frequency ω travels in the x -direction through the vacuum and $\hat{y}$ polarized electric field with amplitude E_0 is given by

$$\vec{E} = E_0 \cos(kx - \omega t) \hat{y}$$

$$\therefore \vec{B} = \frac{\vec{k} \times \vec{E}}{\omega} = \frac{k}{\omega} (\hat{k} \times \vec{E}) = \frac{1}{c} [E_0 \cos(kx - \omega t)] (\hat{x} \times \hat{y}) = \frac{1}{c} E_0 \cos(kx - \omega t) \hat{z}$$

Correct option is (b)



PRACTICE SET

1. Current flowing in a circuit containing a capacitor. To satisfy the continuity equation, $\vec{\nabla} \cdot \vec{J} = -\frac{\partial \rho}{\partial t}$, for charge and current, the current density $\vec{J}$ must be replaced by

(a) $\frac{1}{4\pi} \frac{\partial \vec{D}}{\partial t}$ (b) $\vec{J} - \frac{1}{4\pi} \frac{\partial \vec{D}}{\partial t}$ (c) $\vec{D} + \frac{1}{4\pi} \frac{\partial \vec{J}}{\partial t}$ (d) $\vec{J} + \frac{1}{4\pi} \frac{\partial \vec{D}}{\partial t}$

2. For a medium of electric and magnetic field the relation between $\vec{E}$ and $\vec{B}$ with propagation vector $\vec{k}$

(a) $\vec{E} \times \vec{k} = \omega \vec{B}$ (b) $\vec{E} \times \vec{R} = \omega \times \vec{B}$ (c) $\vec{E} = \frac{\vec{k}}{\omega} \times \vec{B}$ (d) $\vec{B} = \frac{\vec{k} \times \vec{E}}{\omega}$

3. In a free space, which is the correct set of Maxwell's equations in absence of any charges and current densities :

(a) $\vec{\nabla} \cdot \vec{E} = \rho / \epsilon_0, \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$ (b) $\vec{\nabla} \cdot \vec{E} = 0, \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$

$\vec{\nabla} \cdot \vec{B} = 0, \vec{\nabla} \times \vec{B} = \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$

(c) $\vec{\nabla} \cdot \vec{E} = \rho / \epsilon_0, \vec{\nabla} \times \vec{E} = 0$ (d)

$\vec{\nabla} \cdot \vec{B} = 0, \vec{\nabla} \times \vec{B} = \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$

$\vec{\nabla} \cdot \vec{B} = 0, \vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$

(d) $\vec{\nabla} \cdot \vec{E} = 0, \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$

$\vec{\nabla} \cdot \vec{B} = 0, \vec{\nabla} \times \vec{B} = \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$

4. For plane electromagnetic waves varying in time as $e^{-i\omega t}$, the time averaged Poynting vector is:

(a) $\frac{1}{2} R_e (\vec{E} \times \vec{B})$ (b) $R_e (\vec{E} \times \vec{H})$ (c) $\frac{1}{2} R_e (\vec{E} \times \vec{H})$ (d) $\frac{1}{2} (R_e \vec{E}) \times (R_e \vec{H})$

5. Electromagnetic wave in vacuum is given by $\vec{E}(\vec{r}, t) = iE_0 e^{i(\omega t - \vec{k} \cdot \vec{r})}$. Magnetic field $\vec{B}$ is:

(a) $\hat{j} \frac{E_0}{c} \hat{k} \exp[-i(\omega t - \vec{k} \cdot \vec{r})]$

(b) $\hat{j} \frac{E_0}{c} \exp[-i(\omega t - \vec{k} \cdot \vec{r})]$

(c) $\hat{k} \frac{E_0}{c} \exp[-i(\omega t - \vec{k} \cdot \vec{r})]$

(d) $\hat{j} \frac{E_0}{c} \exp[-i(\omega t - \vec{k} \cdot \vec{r})]$

6. The electric field of an electromagnetic wave is given by $\vec{E} = E_0 (3\hat{i} + 4\hat{j}) \cos k(4x - 3y - ct)$. The directions of the magnetic field $\vec{B}$ and the poynting vector $\vec{p}$ are

(a) $-\hat{k}, \frac{4\hat{i} + 3\hat{j}}{5}$

(b) $\hat{k}, \frac{4\hat{i} - 3\hat{j}}{5}$

(c) $\frac{4\hat{i} - 3\hat{j}}{5}, \hat{k}$

(d) $\frac{4\hat{i} - 3\hat{j}}{5}, \frac{4\hat{i} + 3\hat{j}}{5}$

7. For a wave propagating in an air filled metallic rectangular waveguide, the frequency and the wave vector are related as (c is the speed of light and ω_c is the cut off frequency)

(a) $c^2 k^2 = \omega^2 + \omega_c^2$ (b) $c^2 k^2 = \omega^2$ (c) $c^2 k^2 = \omega^2 - \omega_c^2$ (d) $c^2 k^2 = \sqrt{\omega^4 - \omega_c^4}$

8. In a region of space it is found that $\vec{\nabla} \cdot \vec{E} = 0$ but $\vec{\nabla} \times \vec{E} \neq 0$. Which of the following statements is not true?

(a) The charge density must be zero



9. The skin depth of a metal is dependent on the conductivity (σ) of the metal and the angular frequency ω of the incident field. For a metal of high conductivity, which of the following relations is correct? (Assume that $\sigma \gg \epsilon\omega$, where ϵ is the electrical permittivity of the medium).
- (a) $d \propto \sqrt{\sigma/\omega}$ (b) $d \propto \sqrt{1/\sigma\omega}$ (c) $d \propto \sqrt{\sigma\omega}$ (d) $d \propto \sqrt{\omega/\sigma}$
10. The wavelength of red helium-neon laser in air is 6328 Å. What happens to its frequency in glass that has a refractive index of 1.50?
- (a) Increases by a factor of 3 (b) Decreases by a factor of 1.5
(c) Remains the same (d) Decreases by a factor of 0.5

ANSWER KEY

Questions	1	2	3	4	5
Option	(d)	(d)	(d)	(c)	(b)
Questions	6	7	8	9	10
Option	(b)	(a)	(b)	(b)	(c)



Electromagnetic Boundary Values Problem

Boundary Conditions :

When electromagnetic wave exist in more than one medium the field vectors must satisfy certain conditions at the interface of two media. These conditions are known as boundary conditions.

These conditions are

$$\left. \begin{aligned} (1) D_2 \cdot \hat{n} - D_1 \cdot \hat{n} &= \sigma \\ (2) B_2 \cdot \hat{n} &= B_1 \cdot \hat{n} \\ (3) \hat{n} \times \vec{H}_2 - \hat{n} \times \vec{H}_1 &= \vec{k} \\ (4) \vec{E}_2 \cdot \hat{t}_2 &= \vec{E}_1 \cdot \hat{t}_1 \end{aligned} \right\} \dots (1)$$

Special Cases:

(1) **Dielectric-dielectric interface :** In this case, there is no free charge or free current at the interface and we have

$$\left. \begin{aligned} \epsilon_2 E_{2n} &= \epsilon_1 E_{1n}, \quad B_{2n} = B_{1n} \\ \frac{1}{\mu_2} B_{2t} &= \frac{1}{\mu_1} B_{1t}, \quad E_{2t} = E_{1t} \end{aligned} \right\} \dots (2)$$

(n stands for normal component)
(t stands for tangential component)

(2) **Conductor-dielectric interface :** For conducting medium inside field

$$\vec{E}_1 = 0$$

Therefore, $\vec{H}_1$ is also zero. So, both the tangential and normal component of $\vec{H}_1$ and $\vec{E}_1$ are also zero.

$$E_{1n} = E_{1t} = 0 \quad \text{and} \quad H_{1n} = H_{1t} = 0$$

So, in this case, the boundary conditions become

$$\left. \begin{aligned} \epsilon_2 E_{2n} &= \sigma, \quad B_{2n} = B_{1n} = 0 \\ \frac{1}{\mu_2} \hat{n} \times \vec{B}_2 &= \vec{K}, \quad E_{2t} = E_{1t} = 0 \end{aligned} \right\} \dots (3)$$

Fresnel's equations:

p-polarization: If $\vec{E}$ vector of the incident wave is parallel to the plane of incidence is called **p-polarization**.

s-polarization : If $\vec{E}$ vector of the incident wave is perpendicular to the plane of incidence is called **s-polarization**.

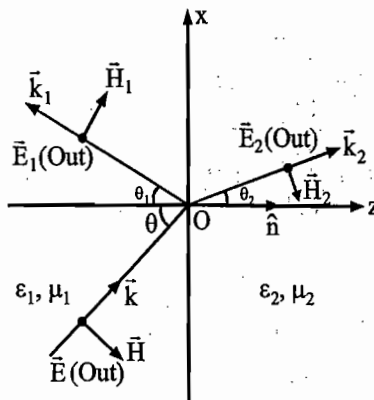


Fig. (1) : Reflection and refraction with $\vec{E}$ -vector normal to the plane of incidence.

Suppose that the $\vec{E}$ -vector of the incident wave lies perpendicular to the plane of incidence (xz -plane) as shown in Fig.(1). As the media on two sides of the interface $z=0$ are isotropic the $\vec{E}$ -vectors of reflected and refracted waves will also be perpendicular to the plane of incidence. Let the incident, reflected and refracted wave are

$$\vec{E} = \vec{E}_0 e^{i(\vec{k} \cdot \vec{r} - \omega t)}, \quad \vec{H} = \frac{\vec{k} \times \vec{E}}{\omega \mu_1} \quad \dots (4.a)$$

$$\vec{E}_1 = \vec{E}_{10} e^{i(\vec{k}_1 \cdot \vec{r} - \omega t)}, \quad \vec{H}_1 = \frac{\vec{k}_1 \times \vec{E}_1}{\omega \mu_1} \quad \dots (4.b)$$

$$\vec{E}_2 = \vec{E}_{20} e^{i(\vec{k}_2 \cdot \vec{r} - \omega t)}, \quad \vec{H}_2 = \frac{\vec{k}_2 \times \vec{E}_2}{\omega \mu_2} \quad \dots (4.c)$$

Since the electric vectors are all parallel to the interface $z=0$, the continuity of the tangential components of $\vec{E}$ -field gives

$$E_0 e^{i(\vec{k} \cdot \vec{r} - \omega t)} + E_{10} e^{i(\vec{k}_1 \cdot \vec{r} - \omega t)} = E_{20} e^{i(\vec{k}_2 \cdot \vec{r} - \omega t)}$$

This condition must be valid for all values of t and for all points on the interface.

$$E_0 + E_{10} = E_{20} \quad \dots (4.d)$$

The continuity of the tangential components of $\vec{H}$ -vectors across the interface $z=0$ requires that

$$-H_0 \cos \theta + H_{10} \cos \theta_1 = -H_{20} \cos \theta_2 \quad \dots (5)$$

Now

$$\vec{H} = \frac{\vec{k} \times \vec{E}}{\omega \mu_1} = \sqrt{\frac{\epsilon_1}{\mu_1}} \cdot \frac{\vec{k} \times \vec{E}}{k}$$

$$\therefore H_0 = \sqrt{\frac{\epsilon_1}{\mu_1}} E_0 \quad \left[\frac{\omega}{k} = \frac{1}{\sqrt{\mu_1 \epsilon_1}} \right]$$

Similarly,

$$H_{10} = \sqrt{\frac{\epsilon_1}{\mu_1}} E_{10} \quad \text{and} \quad H_{20} = \sqrt{\frac{\epsilon_2}{\mu_2}} E_{20}$$

Therefore, equation (5) may be rewritten as

$$(-E_0 \cos \theta + E_{10} \cos \theta) \sqrt{\frac{\epsilon_1}{\mu_1}} = -E_{20} \cos \theta_2 \sqrt{\frac{\epsilon_2}{\mu_2}}, \quad \dots (6)$$

Solving equations (4) and (5) we get

$$r_{\perp} = \frac{E_{10}}{E_0} = \frac{\sqrt{\frac{\epsilon_1}{\mu_1}} \cos \theta - \sqrt{\frac{\epsilon_2}{\mu_2}} \cos \theta_2}{\sqrt{\frac{\epsilon_1}{\mu_1}} \cos \theta + \sqrt{\frac{\epsilon_2}{\mu_2}} \cos \theta_2} \quad \dots (7)$$

$$t_{\perp} = \frac{E_{20}}{E_0} = \frac{2 \sqrt{\frac{\epsilon_1}{\mu_1}} \cos \theta}{\sqrt{\frac{\epsilon_1}{\mu_1}} \cos \theta + \sqrt{\frac{\epsilon_2}{\mu_2}} \cos \theta_2}, \quad \dots (8)$$

For non-magnetic medium, $\mu_1 = \mu_2 = \mu_0$

Therefore, refractive index, $n_1 = \sqrt{\frac{\epsilon_1}{\epsilon_0}}$ and $n_2 = \sqrt{\frac{\epsilon_2}{\epsilon_0}}$

Therefore, the equation (7) and (8) become

$$r_{\perp} = \frac{E_{10}}{E_0} = \frac{n_1 \cos \theta - n_2 \cos \theta_2}{n_1 \cos \theta + n_2 \cos \theta_2} \quad (9)$$

$$t_{\perp} = \frac{E_{20}}{E_0} = \frac{2n_1 \cos \theta}{n_1 \cos \theta + n_2 \cos \theta_2} \quad (10)$$

Using Snell's law

$$\frac{\sin \theta}{\sin \theta_2} = \frac{n_2}{n_1},$$

the Fresnel's equations can be written as

$$r_{\perp} = \frac{E_{10}}{E_0} = \frac{\sin(\theta - \theta_2)}{\sin(\theta + \theta_2)} \quad \dots (11)$$

and

$$t_{\perp} = \frac{E_{20}}{E_0} = -\frac{2 \cos \theta \sin \theta_2}{\sin(\theta + \theta_2)} \quad \dots (12)$$

Case-II: $\vec{E}$ is parallel to the plane of incidence (*p*-polarized wave).

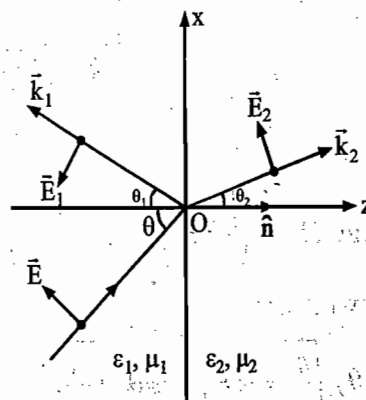


Fig. (3) : Reflection and refraction with $\vec{E}$ vector

Let $\vec{E}$ vector is parallel to the plane of incidence as shown in the fig. (2) and $\vec{E}_1, \vec{E}_2$ and $\vec{E}_3$ and $\vec{H}_1, \vec{H}_2$ and $\vec{H}_3$ are the incident, reflected refracted electric and magnetic field respectively. $\vec{H}_1, \vec{H}_2$ and $\vec{H}_3$ are perpendicular to the incidence plane.

Let the incident, reflected and refracted wave are

$$\vec{E} = \vec{E}_0 e^{j(\vec{k} \cdot \vec{r} - \omega t)}, \quad \vec{H} = \frac{\vec{k} \times \vec{E}}{\omega \mu_1}$$

$$\vec{E}_1 = \vec{E}_{10} e^{j(\vec{k}_1 \cdot \vec{r} - \omega_1 t)}, \quad \vec{H}_1 = \frac{\vec{k}_1 \times \vec{E}_1}{\omega \mu_1}$$

$$\vec{E}_2 = \vec{E}_{20} e^{j(\vec{k}_2 \cdot \vec{r} - \omega_2 t)}, \quad \vec{H}_2 = \frac{\vec{k}_2 \times \vec{E}_2}{\omega \mu_2}$$

Now the continuity of the tangential components of $\vec{E}$ -fields gives

$$E_0 \cos \theta - E_{10} \cos \theta = E_{20} \cos \theta_2, \quad \dots (13)$$

where we use $\theta_1 = \theta$

The continuity of the tangential components of $\vec{H}$ -fields gives

$$H_0 + H_{10} = H_{20} \quad \dots (14)$$

As before,

$$H_0 = \sqrt{\frac{\epsilon_1}{\mu_1}} E_0, \quad H_{10} = \sqrt{\frac{\epsilon_1}{\mu_1}} E_{10} \quad \text{and} \quad H_{20} = \sqrt{\frac{\epsilon_2}{\mu_2}} E_{20}$$

Therefore, equation (14) becomes

$$(E_0 + E_{10}) \sqrt{\frac{\epsilon_1}{\mu_1}} = E_{20} \sqrt{\frac{\epsilon_2}{\mu_2}} \quad \dots (15)$$

Solving equations (14) and (15) we get

$$r_{\parallel} = \frac{E_{10}}{E_0} = \frac{\sqrt{\frac{\epsilon_2}{\mu_2}} \cos \theta - \sqrt{\frac{\epsilon_1}{\mu_1}} \cos \theta_2}{\sqrt{\frac{\epsilon_2}{\mu_2}} \cos \theta + \sqrt{\frac{\epsilon_1}{\mu_1}} \cos \theta_2} \quad \dots (16)$$

and

$$t_{\parallel} = \frac{E_{20}}{E_0} = \frac{2 \sqrt{\frac{\epsilon_1}{\mu_1}} \cos \theta}{\sqrt{\frac{\epsilon_2}{\mu_2}} \cos \theta + \sqrt{\frac{\epsilon_1}{\mu_1}} \cos \theta_2} \quad \dots (17)$$

These two equations are known as Fresnel's equation for the case of polarization in the plane of incidence.

Most dielectrics are essentially nonmagnetic and we can use $\mu_1 = \mu_2 \approx \mu_0$,

$$n_1 = \sqrt{\frac{\epsilon_1}{\epsilon_0}} \quad \text{and} \quad n_2 = \sqrt{\frac{\epsilon_2}{\epsilon_0}}$$

In this case, Fresnel's equations (16) and (17) take the following common form:

$$E_{\parallel} = n_2 \cos \theta - n_1 \cos \theta_2$$



And
$$t_{\parallel} = \frac{E_{20}}{E_0} = \frac{2n_1 \cos \theta}{n_2 \cos \theta + n_1 \cos \theta_2} \quad \dots (19)$$

Using Snell's law

$$\frac{\sin \theta}{\sin \theta_2} = \frac{n_2}{n_1},$$

Fresnel's equation can be rewritten as

$$r_{\parallel} = \frac{E_{10}}{E_0} = \frac{\sin 2\theta - \sin 2\theta_2}{\sin 2\theta + \sin 2\theta_2} = \frac{2 \cos(\theta + \theta_2) \cdot \sin(\theta - \theta_2)}{2 \sin(\theta + \theta_2) \cdot \cos(\theta - \theta_2)} = \frac{\tan(\theta - \theta_2)}{\tan(\theta + \theta_2)} \quad \dots (20)$$

And
$$t_{\parallel} = \frac{E_{20}}{E_0} = \frac{4 \sin \theta_2 \cdot \cos \theta}{\sin 2\theta + \sin 2\theta_2} = \frac{2 \sin \theta_2 \cdot \cos \theta}{\sin(\theta + \theta_2) \cos(\theta - \theta_2)} \quad \dots (21)$$

Reflectance and Transmittance:

We know the intensity of electromagnetic wave is

$$I = \langle s \rangle = \frac{1}{2} \epsilon_1 E_0^2 v_1 \quad \dots (22)$$

where $v_1 = 1/\sqrt{\epsilon_1 \mu_1}$ is the velocity of the wave. Let the incident beam falls on an area A of the interface. Then the cross-sectional area of the incident, reflected and transmitted beams are respectively, $A \cos \theta$, $A \cos \theta_1$ and $A \cos \theta_2$.

$$R = \frac{I_1 A \cos \theta_1}{I A \cos \theta} = \frac{I_1}{I} = \left(\frac{E_{10}}{E_0} \right)^2 \quad \dots (23)$$

[$\because$ Incident and reflected waves are in the same medium]

And
$$T = \frac{I_2 A \cos \theta_2}{I A \cos \theta} = \frac{I_2 \cos \theta_2}{I \cos \theta} = \left(\frac{E_{20}}{E_0} \right)^2 \cdot \frac{\epsilon_2 v_2 \cos \theta_2}{\epsilon_1 v_1 \cos \theta} \quad \dots (24)$$

For nonmagnetic dielectric medium,

$$T = \left(\frac{E_{20}}{E_0} \right)^2 \cdot \frac{n_2 \cos \theta_2}{n_1 \cos \theta} \quad \dots (25)$$

The principle of conservation of energy requires that the rate of energy flowing into area A must be equal to the rate of energy being reflected from it plus the rate of energy being transmitted through A , provided there is no loss at the interface. This requires that

$$R + T = 1. \quad \dots (26)$$

For the case of waves with $\vec{E}$ -vector parallel to the plane of incidence, we can write for R and T , using equations (18) and (19) as

$$R_{\parallel} = \left(\frac{n_2 \cos \theta - n_1 \cos \theta_2}{n_2 \cos \theta + n_1 \cos \theta_2} \right)^2 \quad \dots (27)$$

And

$$T_{\parallel} = \left(\frac{2n_1 \cos \theta}{n_2 \cos \theta + n_1 \cos \theta_2} \right)^2 \cdot \frac{n_2 \cos \theta_2}{n_1 \cos \theta} \quad \dots (28)$$

They are plotted as a function of the angle of incidence in Fig. (3). For waves polarized normal to the plane of incidence one can write, using equations (9) and (10),

and
$$T_{\perp} = \left(\frac{E_{02}}{E_0} \right)^2 \cdot \frac{n_2 \cos \theta_2}{n_1 \cos \theta} = \left(\frac{2n_1 \cos \theta}{n_1 \cos \theta + n_2 \cos \theta_2} \right)^2 \cdot \frac{n_2 \cos \theta_2}{n_1 \cos \theta} \dots (30)$$

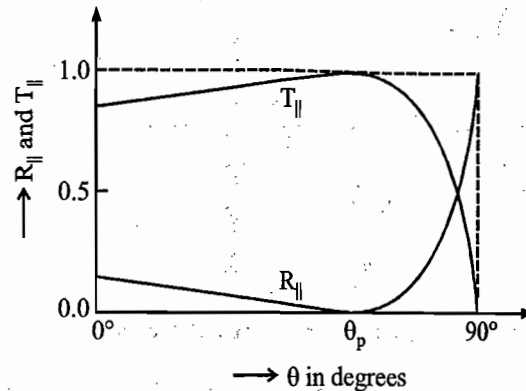


Fig. (3): Reflectance and transmittance as a function of the angle of incidence for $\vec{E}$ -field polarized parallel to the plane of incidence.

A plot of $R_{\perp}$ and $T_{\perp}$ as a function of the angle of incidence is shown in Fig. (4).

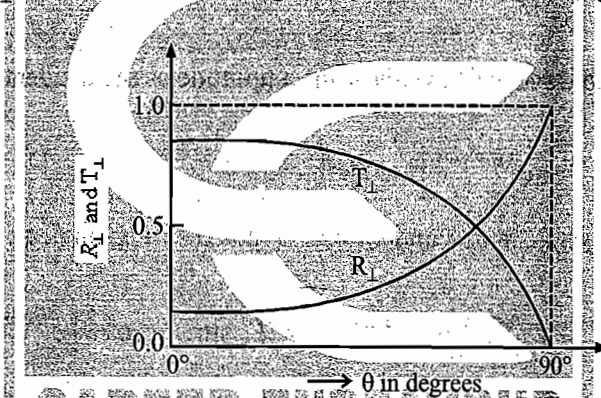


Fig. (4): Reflectance and transmittance as a function of the angle of incidence for $\vec{E}$ -field polarized normal to the plane of incidence.

For normal incidence

$$\theta = \theta_2 = 0$$

And
$$R = R_{\perp} = R_{\parallel} = \left(\frac{n_1 - n_2}{n_1 + n_2} \right)^2$$

And
$$T = T_{\perp} = T_{\parallel} = \frac{4n_1 n_2}{(n_1 + n_2)^2}$$

Propagation of Electromagnetic Waves Between Parallel Conducting Plates:

A structure consisting of highly conducting walls in the form of a long open pipe is used to guide electromagnetic waves from one place to other. Such a structure is called a waveguide. Let two perfectly conducting plates located at $y=0$ and $y=a$ as shown in Fig. (5). The space between the plates is assumed to be free of charges, currents and have permittivity and permeability equal to those of vacuum. Let an electromagnetic wave of frequency ω with wave vector $\vec{k}$ in the yz -plane be incident on the $y=a$ plate with an angle of incidence θ .

$\vec{E}$ vector is perpendicular to the incidence plane.

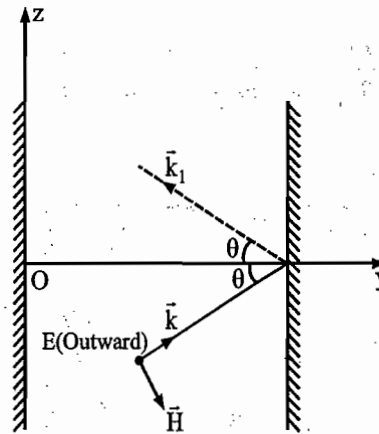


Fig. (5) : Wave propagation between two parallel conducting plates.

Thus, the electric field in the region between the plates for TE waves may be described by

$$\vec{E} = \hat{i}E'_0 e^{j(\vec{k} \cdot \vec{r} - \omega t)} + \hat{i}E'_{10} e^{j(\vec{k}_1 \cdot \vec{r} - \omega t)} \quad \dots (31)$$

Since the plates are assumed to be perfectly conducting the tangential components of $\vec{E}$ -field must vanish at the boundary $y=0$. For this condition to be true for all values of z and t we must have

$$E'_0 = -E'_{10} \quad \dots (32)$$

This indicates that the reflected wave will have the same amplitude as the incident wave but with a phase change of π . Therefore,

$$\vec{E} = \hat{i}E'_0 \left[e^{j(\vec{k} \cdot \vec{r} - \omega t)} - e^{j(\vec{k}_1 \cdot \vec{r} - \omega t)} \right] \quad \dots (33)$$

Now we have $\vec{k} \cdot \vec{r} = k \cos \theta \cdot y + k \sin \theta \cdot z$ and $\vec{k}_1 \cdot \vec{r} = -k \cos \theta \cdot y + k \sin \theta \cdot z$.

Thus,

$$\vec{E} = \hat{i}E'_0 \left[e^{jky \cos \theta} - e^{-jky \cos \theta} \right] \cdot e^{j(kz \sin \theta - \omega t)} = \hat{i}E_0 \sin(ky \cos \theta) e^{j(kz \sin \theta - \omega t)}, \quad \dots (34)$$

where $E_0 = 2jE'_0$. We now apply the boundary condition that the $\vec{E}$ -field must vanish at $y=a$ for all values z and t . This requires that

$$\sin(ka \cos \theta) = 0 \quad \text{or} \quad ka \cos \theta = n\pi, \quad \dots (35a)$$

where n is a positive integer. Thus, for a given k and a , there are a number of discrete directions or modes possible for the propagation of waves between the plates.

For convenience let us rewrite equation(34) as

$$\vec{E} = \hat{i}E_0 \sin(k_c y) e^{j(k_g z - \omega t)}, \quad \dots (35b)$$

where

$$k_c = k \cos \theta \quad \text{and} \quad k_g = k \sin \theta. \quad \dots (36)$$

Equation (36) may now be considered to represent a disturbance propagating along z -axis with an effective wavelength

$$\lambda_g = \frac{2\pi}{k_g} = \frac{2\pi}{k \sin \theta} = \frac{\lambda_0}{\sin \theta}, \quad \dots (37)$$

where $\lambda_0 = 2\pi/k$ is the free space wavelength. The disturbance also varies sinusoidally along y -axis with an effective wavelength

$$\lambda_c = \frac{2\pi}{k_c} = \frac{2\pi}{k \cos \theta} = \frac{\lambda_0}{\cos \theta}. \quad \dots (38)$$

From equations (37) and (38) it follows immediately that



From equation (35a) we can write

$$k_c \cdot a = n\pi \quad \text{or} \quad \lambda_c = \frac{2a}{n} \quad \dots (40)$$

Physical importance of λ_g and λ_c can be derived as follows. Suppose for a given n and a , λ_0 is increased gradually. Equation (39) indicates that if λ_0 becomes greater than λ_c then λ_g becomes imaginary. In this case the exponent $jk_g z$ in equation (35b) becomes negative for $z > 0$, and the wave becomes exponentially damped in z , instead of propagating. Thus, for the waves to propagate between the plates we must have

$$\lambda_0 < \lambda_c = \frac{2a}{n}$$

For this λ_c is called cut off wavelength. It is the longest wavelength that can propagate for a given mode (n value).

λ_g is known as guide wavelength. From equation (39),

$$\lambda_g = \frac{\lambda_0}{\sqrt{1 - \left(\frac{\lambda_0}{\lambda_c}\right)^2}} \quad \dots (41.a)$$

λ_g is greater than corresponding free space wavelength.

Phase Velocity:

$$v_p = \frac{\omega}{k_g} = \frac{\omega}{k \sin \theta} = \frac{c}{\sin \theta} \quad \dots (41.b)$$

The group velocity v_g :

$$v_g = \frac{d\omega}{dk_g}$$

From equations (36) and (40) we can write

$$k_g^2 = k^2 \sin^2 \theta = k^2 (1 - \cos^2 \theta) = k^2 - k_c^2 = k^2 - \frac{n^2 \pi^2}{a^2}$$

Therefore,

$$k_g^2 = \frac{\omega^2}{c^2} - \frac{n^2 \pi^2}{a^2} \quad \left[\because c = \frac{\omega}{k} \right] \quad \dots (43)$$

Differentiating with respect to k_g we get

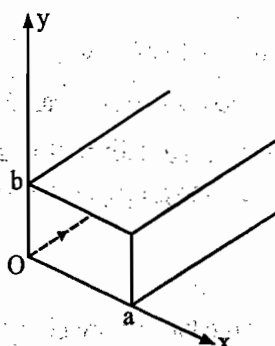
$$2k_g = \frac{2\omega}{c^2} \frac{d\omega}{dk_g} \quad \text{or} \quad \frac{d\omega}{dk_g} = \frac{c^2 k_g}{\omega} = \frac{c^2 \cdot k \sin \theta}{ck} = c \sin \theta. \quad \dots (44)$$

Therefore, group velocity $v_g = c \sin \theta$, which is smaller than the speed of light in vacuum. Moreover, it is interesting to note that

$$v_p \cdot v_g = \frac{c}{\sin \theta} \cdot c \sin \theta = c^2 \quad \dots (45)$$

Rectangular Waveguide :

Let consider a rectangular wave guide of sides a and b as shown in figure



A rectangular waveguide

The space inside the waveguide is vacuum. The electric and magnetic fields are satisfy the homogeneous wave equations

$$\nabla^2 \vec{X} - \frac{1}{c^2} \frac{\partial^2 \vec{X}}{\partial t^2} = 0; \quad (\vec{X} \equiv \vec{E} \text{ or } \vec{H}) \quad \dots (46)$$

If the waves to be propagated are monochromatic with time dependence $e^{j\omega t}$ then equation (46) takes the form

$$\left(\nabla^2 + \frac{\omega^2}{c^2} \right) \vec{X} = 0 \quad \dots (47)$$

and Maxwell's equations

$$\vec{\nabla} \times \vec{H} = \epsilon_0 \frac{\partial \vec{E}}{\partial t} \text{ and } \vec{\nabla} \times \vec{E} = -\mu_0 \frac{\partial \vec{H}}{\partial t} \quad \dots (48)$$

become

$$\vec{\nabla} \times \vec{H} = -j\omega\epsilon_0 \vec{E} \quad \dots (48)$$

And

$$\vec{\nabla} \times \vec{E} = j\omega\mu_0 \vec{H} \quad \dots (49)$$

We seek solutions to equation (47) in the form

$$\vec{X}(x, y, z, t) = \vec{X}_0(x, y) e^{j(k_z z - \omega t)} \quad \dots (50)$$

Equation (48) may be written in component forms as

$$\begin{aligned} \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} &= -j\omega\epsilon_0 E_x; \\ \frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} &= -j\omega\epsilon_0 E_y; \\ \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} &= -j\omega\epsilon_0 E_z. \end{aligned}$$

Using solutions (50) we may write

$$\frac{\partial H_{oz}}{\partial y} - jk_g H_{oy} = -j\omega\epsilon_0 E_{ox}; \quad \dots (51)$$

$$\frac{\partial H_{oy}}{\partial x} - \frac{\partial H_{ox}}{\partial y} = -j\omega\epsilon_0 E_{oz} \quad \dots (53)$$

Similarly, from equation (49) we may get

$$\frac{\partial E_{oz}}{\partial y} - jk_g E_{oy} = j\omega\mu_0 H_{ox}; \quad \dots (54)$$

$$jk_g E_{ox} - \frac{\partial E_{oz}}{\partial x} = j\omega\mu_0 H_{oy}; \quad \dots (55)$$

$$\frac{\partial E_{oy}}{\partial x} - \frac{\partial E_{ox}}{\partial y} = j\omega\mu_0 H_{oz} \cdot R \quad \dots (56)$$

Eliminating H_{oy} from equations (51) and (55) we get

$$E_{ox} = \frac{j}{h^2} \left[k_g \frac{\partial E_{oz}}{\partial x} + \omega\mu_0 \frac{\partial H_{oz}}{\partial y} \right], \quad \dots (57)$$

where

$$h^2 = \frac{\omega^2}{c^2} - k_g^2 \quad \dots (58)$$

Similarly considering other equations appropriately we can express E_{oy} , H_{ox} and H_{oy} in terms of E_{oz} and H_{oz} only.

$$E_{oy} = \frac{j}{h^2} \left[k_g \frac{\partial E_{oz}}{\partial y} - \omega\mu_0 \frac{\partial H_{oz}}{\partial x} \right] \quad \dots (59)$$

$$H_{ox} = \frac{j}{h^2} \left[k_g \frac{\partial H_{oz}}{\partial x} - \omega\epsilon_0 \frac{\partial E_{oz}}{\partial y} \right] \quad \dots (60)$$

$$H_{oy} = \frac{j}{h^2} \left[k_g \frac{\partial H_{oz}}{\partial y} + \omega\epsilon_0 \frac{\partial E_{oz}}{\partial x} \right] \quad \dots (61)$$

Case-I: If $E_{oz} = 0$ then entire field is determined by H_{oz} . For this corresponding waves are called H -waves. Since here only transverse components of electric field exist, it is also called transverse electric or TE waves.

Case-II: If $H_{oz} = 0$ then entire field is determined by E_{oz} . For this corresponding waves are called E -waves. Since here only transverse components of magnetic field exist, it is also called transverse magnetic or TM waves.

TE modes:

In this case, $E_{oz} = 0$ and all other field components may be determined in terms of H_{oz} . From equation (47) we find that the equation satisfied by the z -component of $\vec{H}$ is

$$\left(\nabla^2 + \frac{\omega^2}{c^2} \right) H_z = 0.$$

Taking the z -dependence of H_z of the form $e^{jk_g z}$ this equation can be transformed to

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + h^2 \right) H_{oz} = 0. \quad \dots (62)$$

To solve this equation by the method of separation of variables we assume

$$H_{oz}(x, y) = F_1(x) \cdot F_2(y).$$

Substituting it in equation (62) and dividing throughout by $F_1 F_2$ we get



Let $\frac{1}{F_1} \frac{d^2 F_1}{dx^2} = -k_x^2$ and $\frac{1}{F_2} \frac{d^2 F_2}{dy^2} = -k_y^2$,

where $k_x^2 + k_y^2 = h^2$.

So the most general solution for $H_{oz}(x, y)$ is

$$H_{oz}(x, y) = (A_1 \cos k_x x + A_2 \sin k_x x) (A_3 \cos k_y y + A_4 \sin k_y y), \quad \dots (63)$$

where A_i are arbitrary constant to be evaluated from the given boundary conditions. The boundary conditions are

$$E_y = 0 \text{ at } x = 0 \text{ and } x = a,$$

$$E_x = 0 \text{ at } y = 0 \text{ and } y = b$$

For TE mode $E_{oz} = 0$ and, therefore, from equation (59),

$$E_{oy} \propto \frac{\partial H_{oz}}{\partial x} = 0 \text{ at } x = 0.$$

This condition requires that $A_2 = 0$ in equation (63). From equation (57),

$$E_{ox} \propto \frac{\partial H_{oz}}{\partial y} = 0 \text{ at } y = 0.$$

This condition requires that $A_4 = 0$ in equation (63). Thus,

$$H_{oz}(x, y) = H_0 \cos k_x x \cos k_y y, \quad \dots (64)$$

where $H_0 = A_1 A_3$ and is determined by the introduced field.

Now the condition

$$E_{oy} \propto \frac{\partial H_{oz}}{\partial x} = 0 \text{ at } x = a$$

requires that

$$\sin k_x a = 0 \text{ or } k_x = \frac{m\pi}{a}; m = 0, 1, 2, 3, \dots$$

Similarly, the condition

$$E_{ox} \propto \frac{\partial H_{oz}}{\partial y} = 0 \text{ at } y = b$$

requires that

$$\sin k_y b = 0 \text{ or } k_y = \frac{n\pi}{b}; n = 0, 1, 2, 3, \dots$$

Thus, finally we get

$$H_{oz}(x, y) = H_0 \cos \frac{m\pi x}{a} \cdot \cos \frac{n\pi y}{b} \quad \dots (65)$$

Since H_{oz} is now known other field components can be easily calculated by using the relations (57) - (61).



$$\begin{aligned}
 E_{ox} &= -\frac{j\omega\mu_0}{h^2} H_0 \left(\frac{n\pi}{b}\right) \cos \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \\
 E_{oy} &= \frac{j\omega\mu_0}{h^2} H_0 \left(\frac{m\pi}{a}\right) \sin \frac{m\pi x}{a} \cos \frac{n\pi y}{b} \\
 H_{ox} &= -\frac{jk_g}{h^2} H_0 \left(\frac{m\pi}{a}\right) \sin \frac{m\pi x}{a} \cos \frac{n\pi y}{b} \\
 H_{oy} &= -\frac{jk_g}{h^2} H_0 \left(\frac{n\pi}{b}\right) \cos \frac{m\pi x}{a} \sin \frac{n\pi y}{b}
 \end{aligned} \quad \dots (66)$$

Now

$$\begin{aligned}
 h^2 &= k_x^2 + k_y^2 \quad \text{or} \quad \frac{\omega^2}{c^2} - k_g^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 \\
 \text{or } k_g^2 &= \frac{\omega^2}{c^2} - \left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 \right]
 \end{aligned} \quad \dots (67)$$

If

$$\omega^2 < c^2 \left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 \right]$$

then k_g is imaginary. As the z -dependence of the field quantities is assumed to be of the form $e^{jk_g z}$, the fields will be exponentially damped in z , instead of propagating. Thus, for the waves to propagate along the guide the frequency must be greater than some critical value given by

$$\omega_c = c \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2} \quad \dots (68)$$

This is called cut-off frequency. All frequency components greater than ω_c can propagate through the guide and below it propagation is not possible. The cut-off wavelength is given by

$$\lambda_c = \frac{c}{\omega_c} = \frac{2}{\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}} \quad \dots (69)$$

Propagation is possible for wavelength $\lambda < \lambda_c$.

Now that the values of ω_c or λ_c depend on the transverse dimensions of the waveguide and also on the mode (m, n) of propagation. TE mode for arbitrary values of m and n is denoted by $(TE)_{mn}$. A mode for which λ_c is highest is called a dominant mode. Note that $(TE)_{00}$ mode is not possible because for $m = n = 0$, $H_{oz} = H_0$ and all other field components vanish; further, the condition $\vec{\nabla} \cdot \vec{H} = 0$ requires that $ik_g H_z = 0$ or $H_z = 0$.

From (67) and (68) we can write

$$k_g^2 = \frac{\omega^2}{c^2} - \frac{\omega_c^2}{c^2} \quad \dots (70)$$

The quantity $\frac{2\pi}{k_g} = \lambda_g$ is called guide wavelength. $\lambda_0 = \frac{c}{\omega/2\pi}$ is the free space wavelength. So we can write from (70)



$$\frac{1}{\lambda_g^2} = \frac{1}{\lambda_c^2} - \frac{1}{\lambda_0^2} \text{ or } \lambda_g = \frac{\lambda_0}{\sqrt{1 - \left(\frac{\lambda_0}{\lambda_c}\right)^2}} \quad \dots (71)$$

Thus, guide wavelength is greater than corresponding free space wavelength. The phase velocity of the wave along z -direction is

$$v_p = \frac{\omega}{k_g} = \frac{\omega}{\sqrt{\frac{\omega^2}{c^2} - \frac{\omega_c^2}{c^2}}} = \frac{c}{\sqrt{1 - \left(\frac{\lambda_0}{\lambda_c}\right)^2}} \quad \dots (72)$$

Obviously phase velocity of the wave is greater than the velocity of light in free space. This does not violate the principle of relativity because the group velocity with which energy propagates along the guide is always smaller than c . Using (70) we can show that the group velocity

$$v_g = \frac{d\omega}{dk_g} = c \sqrt{1 - \left(\frac{\lambda_0}{\lambda_c}\right)^2} \quad \dots (73)$$

It is interesting to note that

$$v_p v_g = c^2. \quad \dots (74)$$

Power flow associated with TE waves

The time average power flowing through the wavelength along z -axis is given by

$$\begin{aligned} P &= \int_0^a \int_0^b \frac{1}{2} \operatorname{Re} (\vec{E} \times \vec{H}^*) \cdot \hat{k} dx dy = \frac{1}{2} \operatorname{Re} \int_0^a \int_0^b (E_x H_y^* - E_y H_x^*) dx dy \\ &= \frac{1}{2} \operatorname{Re} \int_0^a \int_0^b (E_{ox} H_{oy}^* - E_{oy} H_{ox}^*) dx dy = \frac{1}{2} \frac{\omega \mu_0 k_g}{h^4} H_0^2 \int_0^a \int_0^b \left[\left(\frac{n\pi}{b} \right)^2 \cos^2 \frac{n\pi x}{a} \sin^2 \frac{n\pi y}{b} \right. \\ &\quad \left. + \left(\frac{m\pi}{a} \right)^2 \sin^2 \frac{m\pi x}{a} \cos^2 \frac{n\pi y}{b} \right] dx dy \\ &= \frac{H_0^2 \omega \mu_0 k_g}{2h^4} \cdot \frac{a}{2} \cdot \frac{b}{2} \cdot \left[\left(\frac{m\pi}{a} \right)^2 + \left(\frac{n\pi}{b} \right)^2 \right] = \frac{ab}{8} \cdot \frac{\omega \mu_0 k_g}{h^2} \cdot H_0^2 \left[\because h^2 = \left(\frac{m\pi}{a} \right)^2 + \left(\frac{n\pi}{b} \right)^2 \right] \quad \dots (75) \end{aligned}$$

For either $m = 0$ or $n = 0$, we have

$$P = \frac{ab}{4} \cdot \frac{\omega \mu_0 k_g}{h^2} \cdot H_0^2.$$

Thus, for a particular waveguide power transmission will be greater for larger H_0 . A waveguide, however, cannot transit any arbitrary large power. The maximum value is limited by the puncture of the dielectric filling the waveguide.

TM modes

In this case, the waveguide contains no z -component of magnetic field, i.e., $H_{oz} = 0$ but $E_{oz} \neq 0$. Here we start from the equation satisfied by E_z and proceeding as before we can write for the most general solution of E_{oz} as

$$E_{oz}(x, y) = (A_1 \cos k_x x + A_2 \sin k_x x) (A_3 \cos k_y y + A_4 \sin k_y y).$$

To find the constant A_i we apply the boundary conditions $E_{oz} = 0$ at $x = 0, x = a, y = 0, y = b$.



$$E_{oz} = 0 \text{ at } x = 0 \text{ requires that } A_1 = 0.$$

$$E_{oz} = 0 \text{ at } y = 0 \text{ requires that } A_3 = 0.$$

Thus,

$$E_{oz} = E_0 \sin k_x x \sin k_y y,$$

where $E_0 = A_2 A_4$ is determined by introduced field.

Now the condition $E_{oz} = 0$ at $x = a$ requires that

$$\sin k_x a = 0 \text{ or } k_x = \frac{m\pi}{a}; \quad m = 0, 1, 2, \dots$$

Similarly, the condition $E_{oz} = 0$ at $y = b$ implies that

$$\sin k_y b = 0 \text{ or } k_y = \frac{n\pi}{b}; \quad n = 0, 1, 2, \dots$$

Thus, finally we get

$$E_{oz}(x, y) = E_0 \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \quad \dots (76)$$

Now as E_{oz} is known other field components can be easily calculated in terms of it.

$$\begin{aligned} E_{ox} &= \frac{jk_z}{h^2} \frac{\partial E_{oz}}{\partial x} = \frac{jk_z}{h^2} E_0 \left(\frac{m\pi}{a} \right) \cos \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \\ E_{oy} &= \frac{jk_z}{h^2} \frac{\partial E_{oz}}{\partial y} = \frac{jk_z}{h^2} E_0 \left(\frac{n\pi}{b} \right) \sin \frac{m\pi x}{a} \cos \frac{n\pi y}{b} \\ H_{ox} &= \frac{j\omega\epsilon_0}{h^2} \frac{\partial E_{oz}}{\partial y} = \frac{j\omega\epsilon_0}{h^2} E_0 \left(\frac{n\pi}{b} \right) \sin \frac{m\pi x}{a} \cos \frac{n\pi y}{b} \\ H_{oy} &= \frac{j\omega\epsilon_0}{h^2} \frac{\partial E_{oz}}{\partial x} = \frac{j\omega\epsilon_0}{h^2} E_0 \left(\frac{m\pi}{a} \right) \cos \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \quad \dots (77) \end{aligned}$$

As before we can calculate the cut off wavelength, which is the same as given by (69). In TM mode all the field components vanish for both m and n zero or any one of them zero. Thus, $(TM)_{00}$, $(TM)_{10}$ or $(TM)_{01}$ modes are not possible. For TM mode the longest wavelength available for transmission through the guide is

$$(\lambda_c)_{11} = \frac{2}{\sqrt{\left(\frac{1}{a}\right)^2 + \left(\frac{1}{b}\right)^2}},$$

which is definitely smaller than that for TE mode, where the longest wavelength is

$$(\lambda_c)_{10} = \frac{2}{\sqrt{\left(\frac{1}{a}\right)^2}} \quad (\text{assuming } a > b)$$



This result shows that the frequency band available for transmission through the waveguide is smaller in TM mode than that in TE mode. In this respect, TE mode is superior to TM mode of propagation.

Power flow associated with TM modes

Time average power transmitted through the waveguide in TM modes is given by

$$P = \int_0^a \int_0^b \frac{1}{2} \operatorname{Re}(\vec{E} \times \vec{H}^*) \cdot \hat{k} dx dy.$$

As before,

$$P = \frac{1}{2} \operatorname{Re} \int_0^a \int_0^b (E_{ox} H_{oy}^* - E_{oy} H_{ox}^*) dx dy.$$

Using the field components given by (76) we can get

$$P = \frac{ab}{8} \cdot \frac{\omega \epsilon_0 k_g}{h^2} \cdot E_0^2.$$

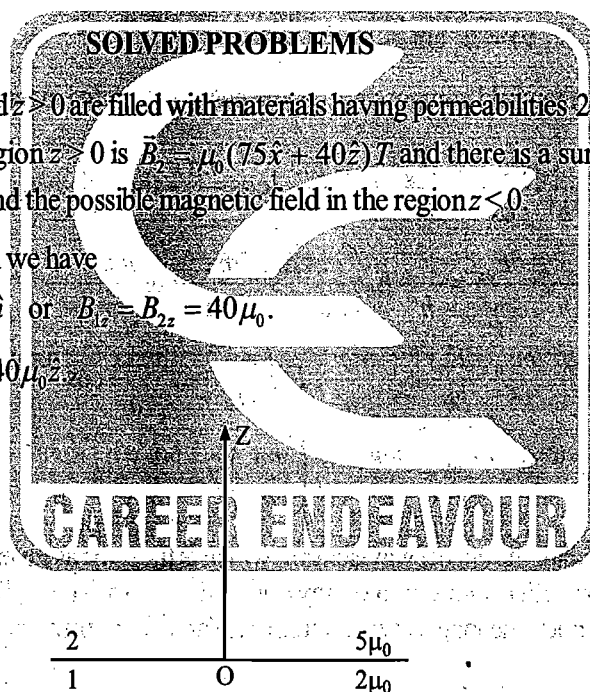
SOLVED PROBLEMS

- The regions of space $z < 0$ and $z > 0$ are filled with materials having permeabilities $2\mu_0$ and $5\mu_0$, respectively. The magnetic field in the region $z > 0$ is $\vec{B}_2 = \mu_0(75\hat{x} + 40\hat{z})T$ and there is a surface current distribution $\vec{K} = -10\hat{y} A/m$ at $z = 0$. Find the possible magnetic field in the region $z < 0$.

Soln. From the boundary condition we have

$$\vec{B}_2 \cdot \hat{n} = \vec{B}_1 \cdot \hat{n} \quad \text{or} \quad B_{1z} = B_{2z} = 40\mu_0.$$

$$\therefore (\vec{B}_1)_{\text{normal}} = 40\mu_0 \hat{z}$$



Also,

$$\hat{n} \times \vec{H}_2 - \hat{n} \times \vec{H}_1 = \vec{K}$$

$$\text{or} \quad \hat{z} \times \frac{\vec{B}_2}{5\mu_0} - \hat{z} \times \frac{\vec{B}_1}{2\mu_0} = -10\hat{y}$$

$$\text{or} \quad \hat{z} \times (15\hat{x} + 8\hat{z}) - \hat{z} \times \frac{\vec{B}_1}{2\mu_0} = -10\hat{y}$$

$$\text{or} \quad \hat{z} \times \frac{\vec{B}_1}{2\mu_0} = 25\hat{y} = 25\hat{z} \times \hat{x}$$

$$\text{or} \quad (\vec{B}_1)_{\text{tangential}} = 50\mu_0 \hat{x}$$



$$\vec{E} = \hat{x} 10 \cos(kz - \omega t) V.m^{-1}$$

is incident from air on a dielectric occupying the region $z \geq 0$. Assuming that the permittivity of the medium is $4\epsilon_0$ and permeability μ_0 determine the electric field of the reflected and transmitted waves.

Soln. Refractive index of the medium is

$$n_2 = \sqrt{\frac{\mu}{\mu_0} \cdot \frac{\epsilon}{\epsilon_0}} = \sqrt{4} = 2.$$

Therefore, amplitude reflection coefficient will have the magnitude

$$|r| = \frac{n_2 - n_1}{n_2 + n_1} = \frac{2 - 1}{2 + 1} = \frac{1}{3}.$$

Now considering the phase reversal of the $\vec{E}$ -field in the reflected wave we can write for the reflected wave (propagating in the negative z -direction),

$$\vec{E}_1 = -\hat{x} \frac{10}{3} \cos(kz + \omega t) V.m^{-1}.$$

The amplitude transmission coefficient

$$t_{12} = \frac{2n_1}{n_2 + n_1} = \frac{2}{3}.$$

Therefore, the transmitted wave may be represented by

$$\vec{E}_2 = \hat{x} \frac{20}{3} \cos(kz - \omega t) V.m^{-1}$$

3. Consider a perfectly conducting rectangular waveguide infinitely long along z -axis with a width 2 cm along x -axis and 1 cm along y -axis. An EM wave whose free space wavelength is 3 cm propagating along the guide in TE_{10} mode. Find the guide wavelength and the speed of propagation of electromagnetic energy along the guide.

Soln. For TE_{10} mode the cut-off wavelength $\lambda_c = 2a = 4$ cm

The free space wavelength $\lambda_0 = 3$ cm

The guide wavelength

$$\lambda_g = \frac{\lambda_0}{\sqrt{1 - \left(\frac{\lambda_0}{\lambda_c}\right)^2}} = \frac{3}{\sqrt{1 - \left(\frac{3}{4}\right)^2}} \text{ cm} = 4.5 \text{ cm}$$

The speed of propagation of electromagnetic energy

= group velocity of the waves

$$= c \sqrt{1 - \left(\frac{\lambda_0}{\lambda_c}\right)^2} = c \sqrt{1 - \left(\frac{3}{4}\right)^2} = 0.66 c,$$

where c is the speed of light in free space.

4. Consider three dielectric media A , B and C forming two interfaces $A-B$ and $B-C$. The refractive indices of the media are n_1 , n_2 and n_3 , respectively. An EM wave from medium A is incident on the medium B normally. Show that for the amplitude reflection coefficients at $A-B$ and $B-C$ interfaces will be equal if $n_2 = \sqrt{n_1 n_3}$.

Soln. For equality of amplitudes reflection coefficients we must have

$$\frac{n_2 - n_1}{n_2 + n_1} = \frac{n_3 - n_2}{n_3 + n_2} \text{ or } n_2 = \sqrt{n_1 n_3}.$$



5. (i) Consider a waveguide of square cross-section having side a . It is to transmit an EM wave of wavelength λ in TE_{10} mode but not in the TE_{11} or TM_{11} modes. Find the limiting range of values of a .
- (ii) Consider a waveguide of rectangular cross-section having dimensions $2.3 \text{ cm} \times 1.0 \text{ cm}$. Find the TE modes that can propagate in this waveguide, if the driving frequency is $1.7 \times 10^{10} \text{ Hz}$. If you are to excite only the TE_{10} mode what frequency range you can use? What is the corresponding free space wavelength range?

Soln. (i) The cut-off wavelength

$$\lambda_{c,mn} = \frac{2}{\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{a}\right)^2}}$$

$$\therefore \lambda_{c,10} = 2a \text{ and } \lambda_{c,11} = \sqrt{2}a.$$

So for the propagation of EM wave in TE_{10} mode its wavelength

$$\lambda < \lambda_{c,10} \text{ or } \lambda < 2a \text{ or } \lambda/2 < a$$

and for the non-propagation of TE_{11} or TM_{11} modes

$$\lambda > \lambda_{c,11} \text{ or } \lambda > \sqrt{2}a \text{ or } a < \lambda/\sqrt{2}$$

Therefore, the required range of values of a is

$$\frac{\lambda}{2} < a < \frac{\lambda}{\sqrt{2}}$$

$$(ii) \quad \nu_{c,mn} = \frac{\omega_{c,mn}}{2\pi} = \frac{c}{2\pi} \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}$$

$$\therefore \nu_{c,10} = \frac{c}{2a} = \frac{3 \times 10^8}{2 \times 2.3 \times 10^{-2}} \text{ Hz} = 0.65 \times 10^{10} \text{ Hz}$$

$$\nu_{c,20} = 2 \times \frac{c}{2a} = 1.30 \times 10^{10} \text{ Hz}, \nu_{c,30} = 3 \times \frac{c}{2a} = 1.95 \times 10^{10} \text{ Hz}$$

$$\nu_{c,01} = \frac{c}{2b} = 1.5 \times 10^{10} \text{ Hz}, \nu_{c,02} = 2 \times \frac{c}{2b} = 3.0 \times 10^{10} \text{ Hz}$$

$$\nu_{c,11} = \frac{c}{2} \sqrt{\frac{1}{a^2} + \frac{1}{b^2}} = 1.63 \times 10^{10} \text{ Hz}$$

As the driving frequency is greater than $\nu_{c,10}$, $\nu_{c,20}$, $\nu_{c,01}$ and $\nu_{c,11}$ these modes can propagate in the waveguide.

In order to excite only the TE_{10} mode exciting frequency ν must be greater than $\nu_{c,10}$ but smaller than the critical frequency of the adjacent mode. So we must have $\nu_{c,10} < \nu < \nu_{c,20}$. As $\lambda = c/\nu$, so

$$\nu_{c,10} = \frac{c}{\lambda_{c,10}} = \frac{c}{2a} \text{ and } \lambda_{c,20} = \frac{c}{\nu_{c,20}} = a.$$

Therefore, we must have

$$\frac{c}{\lambda_{c,10}} < \frac{c}{\lambda} < \frac{c}{\lambda_{c,20}} \text{ or } \lambda_{c,20} < \lambda < \lambda_{c,10} \text{ or } a < \lambda < 2a \text{ or } 2.3 \text{ cm} < \lambda < 4.6 \text{ cm}$$

6. In a rectangular waveguide having dimensions $a \times b$, show that the cut-off wavelength for the lowest TE mode is higher than that of the lowest TM mode by a factor $\sqrt{1 + a^2/b^2}$.



$$\lambda_{c,mn} = \frac{2}{\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}}$$

For the lowest TE mode $m = 1, n = 0$. Therefore, $\lambda_{c,10}^{TE} = 2a$.

For the lowest TM mode $m = 1, n = 0$. Therefore,

$$\lambda_{c,11}^{TM} = \frac{2}{\sqrt{\frac{1}{a^2} + \frac{1}{b^2}}} = \frac{2a}{\sqrt{1 + a^2/b^2}} \quad \text{or} \quad \lambda_{c,10}^{TE} = \left(\sqrt{1 + \frac{a^2}{b^2}} \right) \lambda_{c,11}^{TM} \quad (\text{proved}).$$

7. Electromagnetic waves are propagating along a hollow, metallic waveguide whose cross-section is a square of side W . The minimum frequency of the electromagnetic waves is :

(a) c/W (b) $2c/W$ (c) $\pi c/W$ (d) $\sqrt{2} \pi c/W$ [GATE 2003]

Soln. We the cut-off frequency

$$\omega = c \sqrt{\left(\frac{\pi m}{a}\right)^2 + \left(\frac{\pi n}{b}\right)^2}$$

For square cross-section, $a = b = W$

$$\text{Therefore, the minimum } \omega \text{ is } = c \sqrt{\left(\frac{\pi}{W}\right)^2 + \left(\frac{\pi}{W}\right)^2} = \left(\frac{\pi c}{W}\right)$$

Correct option is (c)

8. A plane electromagnetic wave travelling in vacuum is incident normally on a non magnetic, non-absorbing medium of refractive index n . The incident (E_i), reflected (E_r) and transmitted (E_t) electric fields are given as,

[GATE 2005]

$$E_i = E \exp[i(kz - \omega t)], E_r = E_0 \exp[i(k_r z - \omega t)], E_t = E_{0t} \exp[i(k_t z - \omega t)]$$

If $E = 2 \text{ V/m}$ and $n = 1.5$ then the application of appropriate boundary conditions leads to :

$$(a) E_{0r} = -\frac{3}{5} \text{ V/m}, E_{0t} = \frac{7}{5} \text{ V/m}$$

$$(b) E_{0r} = -\frac{1}{5} \text{ V/m}, E_{0t} = \frac{8}{5} \text{ V/m}$$

$$(c) E_{0r} = -\frac{2}{5} \text{ V/m}, E_{0t} = \frac{8}{5} \text{ V/m}$$

$$(d) E_{0r} = \frac{4}{5} \text{ V/m}, E_{0t} = \frac{6}{5} \text{ V/m}$$

Soln. We have, $E_i = E_0 e^{i(\vec{k} \cdot \vec{r} - \omega t)}$

$$E_0 = E_{0r} e^{i(k_r r - \omega t)}$$

$$E_t = E_0 + e^{i(\vec{k}_t \cdot \vec{r} - \omega t)}$$

We know for normal incidence,

$$r = \frac{E_{0r}}{E_0} = \frac{n_1 - n_2}{n_1 + n_2}$$

Here, $n_1 = 1, n_2 = 1.5$

$$\frac{E_{0r}}{E_0} = \frac{-0.5}{2.5}$$



$$t = \frac{E_{0t}}{E_0} = \frac{2n_1}{n_1 + n_2} = \frac{2 \times 1}{1 + 1.5} = \frac{2}{2.5}$$

$$\Rightarrow E_{0t} = \frac{2 \times 2}{2.5} = \frac{4}{2.5} = \frac{8}{5} \text{ V}$$

Correct option is (c)

9. A plane electromagnetic wave traveling in free space is incident normally on a glass plate of refractive index $\frac{3}{2}$. If there is no absorption by the glass, its reflectivity is [GATE 2012]
 (a) 4% (b) 16% (c) 20% (d) 50%

Soln. We know that the reflectivity for normal incident is

$$R = \left(\frac{n_1 - n_2}{n_1 + n_2} \right)^2 = \left(\frac{1 - 1.5}{1 + 1.5} \right)^2 = \left(\frac{0.5}{2.5} \right)^2 = \left(\frac{1}{5} \right)^2 = \frac{1}{25} = 0.04 = 4\%$$

Correct option is (a)

Common Data for Q. 10 and Q. 11

[GATE 2007]

The Fresnel relations between the amplitudes of incident and reflected electromagnetic waves at an interface between air and a dielectric of refractive index μ , are

$$E_{\parallel}^{\text{reflected}} = \frac{\cos r - \mu \cos i}{\cos r + \mu \cos i} E_{\parallel}^{\text{incident}} \quad \text{and} \quad E_{\perp}^{\text{reflected}} = \frac{\cos r - \mu \cos i}{\cos r + \mu \cos i} E_{\perp}^{\text{incident}}$$

The subscripts $\parallel$ and $\perp$ refer to polarization, parallel to the plane of incidence respectively. Here, i and r are the angles of incidence and refraction respectively.

10. The condition for the reflected ray to be completely polarized is
 (a) $\mu \cos i = \cos r$ (b) $\cos i = \mu \cos r$ (c) $\mu \cos i = -\cos r$ (d) $\cos i = -\mu \cos r$

Soln. When the incident angle i is become (i_p) Brewster angle then the reflected light is completely polarized.

$$\therefore E_{\parallel}^{\text{reflected}} = 0$$

$$\therefore \cos r - \mu \cos i = 0$$

$$\Rightarrow \mu \cos i = \cos r$$

Correct option is (a)

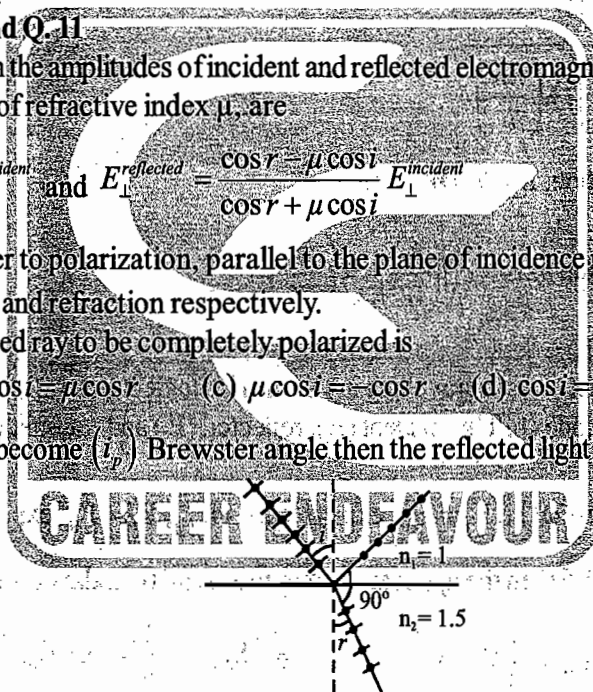
11. For normal incident at an air-glass interface with $\mu = 1.5$ the fraction of energy reflected is given by
 (a) 0.40 (b) 0.20 (c) 0.16 (d) 0.04

Soln. We know that

$$R = \left(\frac{n_1 - n_2}{n_1 + n_2} \right)^2 \quad \text{for normal incidence}$$

$$= \left(\frac{1 - 1.5}{1 + 1.5} \right)^2 = \left(\frac{0.5}{2.5} \right)^2 = \frac{1}{25} = 0.04$$

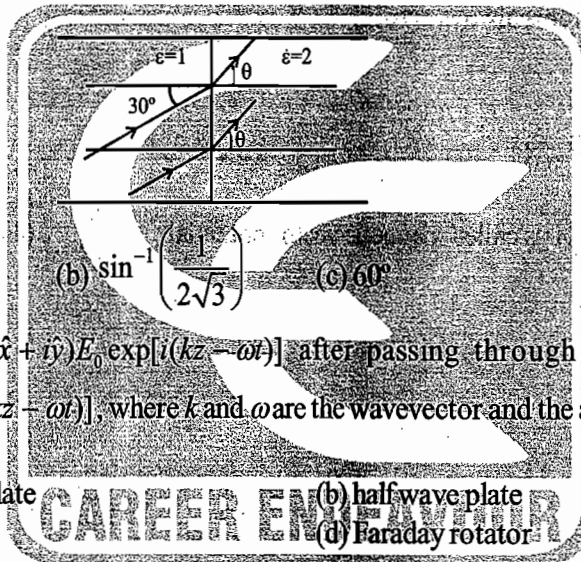
Correct option is (d)





PRACTICE SET

- An electromagnetic field is described by the scalar potential $\phi = ay$ and vector potential $\vec{A} = \hat{k}t$ where a is a constant. The electric field $\vec{E}$ is:
 - Linearly polarized
 - Circularly polarized
 - Elliptically polarized
 - Unpolarized.
- An electromagnetic wave is elliptically polarized in the X-Y plane. The direction of the Poynting vector is in the
 - Z-direction
 - X-direction
 - Y-direction
 - XY-plane
- Polarisation can be achieved by
 - Beam splitter
 - Grating
 - Plane Mirror
 - None of these
- A large slab consists of two materials with dielectric constants 1 and 3 divided by a plane interface as shown in the figure. An electric field exists in this slab, which uniformly makes an angle of 30° with the normal to the interface on the left side of the ($\epsilon = 1$). The angle the field makes on the other side of the interface is:



- $\tan^{-1}(3)$
 - $\sin^{-1}\left(\frac{1}{2\sqrt{3}}\right)$
 - 60°
 - $\tan^{-1}(1/3)$
- A plane wave $(\hat{x} + i\hat{y})E_0 \exp[i(kz - \omega t)]$ after passing through an optical element emerges as $(\hat{x} - i\hat{y})E_0 \exp[i(kz - \omega t)]$, where k and ω are the wavevector and the angular frequency, respectively. The optical element is
 - quarter wave plate
 - half wave plate
 - polarizer
 - Faraday rotator
 - A monochromatic plane wave (wavelength = 600 nm) $E_0 \exp[i(kz - \omega t)]$ is incident normally on a diffraction grating giving rise to a plane wave $E_1 \exp[i(\vec{k}_1 \cdot \vec{r} - \omega t)]$ in the first order of diffraction. Here

$E_1 < E_0$ and $\vec{k}_1 = |\vec{k}_1| \left[\frac{1}{2} \hat{x} + \frac{\sqrt{3}}{2} \hat{z} \right]$. The period (in μm) of the diffraction grating is _____ (upto one decimal place)

ANSWER KEY						
Questions	1	2	3	4	5	6
Option	(a)	(a)	(a)	(c)	(a)	(1.2)

Gauge & Lorentz Transformations

Potential Formation:

From Maxwell's equation,

$$\vec{\nabla} \cdot \vec{B} = 0$$

Since divergence of curl of a vector is always zero,

$$\vec{\nabla} \cdot (\vec{\nabla} \times \vec{A}) = 0$$

$\therefore$

$$\vec{\nabla} \cdot \vec{B} = 0 = \vec{\nabla} \cdot (\vec{\nabla} \times \vec{A})$$

Therefore, $\boxed{\vec{B} = \vec{\nabla} \times \vec{A}}$ where, $\vec{A}$ is called magnetic vector potential. (1)

Also we know,

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$\Rightarrow$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial}{\partial t} (\vec{\nabla} \times \vec{A})$$

$\Rightarrow$

$$\vec{\nabla} \times \left(\vec{E} + \frac{\partial \vec{A}}{\partial t} \right) = 0$$

Since curl of grad of a scalar function is always zero

$\therefore$

$$\vec{\nabla} \times \left(\vec{E} + \frac{\partial \vec{A}}{\partial t} \right) = 0 = -\vec{\nabla} \times \vec{\nabla} \phi$$

$\Rightarrow$

$$\vec{E} + \frac{\partial \vec{A}}{\partial t} = -\vec{\nabla} \phi$$

$\Rightarrow$

$$\boxed{\vec{E} = -\vec{\nabla} \phi - \frac{\partial \vec{A}}{\partial t}} \quad \dots (2)$$

Gauge Transformation:

We know that,

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

$$\vec{E} = -\vec{\nabla} \phi - \frac{\partial \vec{A}}{\partial t}$$



$$\begin{aligned}\therefore \vec{B}' &= \vec{\nabla} \times \vec{A}' = \vec{\nabla} \times (\vec{A} + \vec{\nabla} \psi) \\ &= \vec{\nabla} \times \vec{A} + \vec{\nabla} \times \vec{\nabla} \psi = \vec{\nabla} \times \vec{A} + 0 = \vec{B}\end{aligned}$$

$$\begin{aligned}\therefore \vec{E}' &= -\vec{\nabla} \phi' - \frac{\partial \vec{A}'}{\partial t} \\ &= -\vec{\nabla} \phi + \vec{\nabla} \frac{\partial \psi}{\partial t} - \frac{\partial \vec{A}}{\partial t} - \frac{\partial}{\partial t} (\vec{\nabla} \psi) \\ &= -\vec{\nabla} \phi + \vec{\nabla} \frac{\partial \psi}{\partial t} - \frac{\partial \vec{A}}{\partial t} - \vec{\nabla} \left(\frac{\partial \psi}{\partial t} \right) \\ &= -\vec{\nabla} \phi - \frac{\partial \vec{A}}{\partial t} = \vec{E}\end{aligned}$$

Therefore, under this transformation $\vec{B}$ and $\vec{E}$ field remain unchanged.

$$\begin{aligned}\vec{A}' &= \vec{A} + \vec{\nabla} \psi \\ \phi' &= \phi - \frac{\partial \psi}{\partial t}\end{aligned}$$

is called gauge transformation.

... (3)

10.2 : Lorentz Gauge:

We know that

$$\begin{aligned}\vec{\nabla} \cdot \vec{D} &= \rho \\ \Rightarrow \epsilon (\vec{\nabla} \cdot \vec{E}) &= \rho \\ \Rightarrow -\epsilon \left(\vec{\nabla} \cdot \left(\vec{\nabla} \phi + \frac{\partial \vec{A}}{\partial t} \right) \right) &= \rho \\ \Rightarrow \nabla^2 \phi + \frac{\partial}{\partial t} (\vec{\nabla} \cdot \vec{A}) &= -\frac{\rho}{\epsilon}\end{aligned}$$

... (4)

Also,

$$\vec{\nabla} \times \vec{B} = \mu \vec{J} + \epsilon \mu \frac{\partial \vec{E}}{\partial t}$$

$$\begin{aligned}\Rightarrow \vec{\nabla} \times (\vec{\nabla} \times \vec{A}) &= \mu \vec{J} + \epsilon \mu \left(-\frac{\partial^2 \vec{A}}{\partial t^2} - \frac{\partial}{\partial t} \vec{\nabla} \phi \right) \\ \Rightarrow \vec{\nabla} (\vec{\nabla} \cdot \vec{A}) - \vec{\nabla}^2 \vec{A} &= \mu \vec{J} - \epsilon \mu \frac{\partial^2 \vec{A}}{\partial t^2} - \epsilon \mu \frac{\partial^2 \vec{A}}{\partial t^2} - \epsilon \mu \frac{\partial}{\partial t} \vec{\nabla} \phi \\ \Rightarrow \vec{\nabla} \vec{A} - \mu \epsilon \frac{\partial^2 \vec{A}}{\partial t^2} &= -\mu \vec{J} + \vec{\nabla} \left[\vec{\nabla} \cdot \vec{A} + \mu \epsilon \frac{\partial \phi}{\partial t} \right]\end{aligned}$$

... (5)

But it is very difficult to find out $\vec{A}$ and ϕ from above two equations. We can remove this difficulty by taking

$$\vec{\nabla} \cdot \vec{A} + \mu \epsilon \frac{\partial \phi}{\partial t} = 0$$

is called Lorentz Gauge condition.

... (6)



$$\nabla^2 \phi - \mu\epsilon \frac{\partial^2 \phi}{\partial t^2} = -\frac{\rho}{\epsilon} \quad \dots (7)$$

$$\nabla^2 \vec{A} - \mu\epsilon \frac{\partial^2 \vec{A}}{\partial t^2} = -\mu \vec{J} \quad \dots (8)$$

Therefore, under Lorentz Gauge conditions ϕ and $\vec{A}$ satisfy the inhomogeneous wave equation with a source term on the right. So, we can use the Lorentz Gauge exclusively to solve electrodynamic problem.

Coulomb Gauge:

In this gauge we take $\vec{\nabla} \cdot \vec{A} = 0$, then the equations (7) and (8)

$$\vec{\nabla}^2 \phi = -\frac{\rho}{\epsilon_0} \quad \dots (9)$$

$$\vec{\nabla}^2 \vec{A} - \mu\epsilon \frac{\partial^2 \vec{A}}{\partial t^2} = -\mu \vec{J} \quad (10)$$

Equation (10) we can solve easily

$$\phi(r, t) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(r', t)}{|r - r'|} dv$$

But it is difficult to find $\vec{A}$ by solving equation (10). The Coulomb Gauge is often utilized in cases where there are no charge or current distributions i.e. $\rho = 0$, $\vec{J} = 0$ therefore, one takes $\phi = 0$ and $\vec{A}$ satisfy the equation

$$\vec{\nabla}^2 \vec{A} - \mu\epsilon \frac{\partial^2 \vec{A}}{\partial t^2} = 0$$

In this case the fields are derivable from a single potential

$$\vec{B} = \vec{\nabla} \times \vec{A} \text{ and } \vec{E} = -\frac{\partial \vec{A}}{\partial t}$$

10.3. Radiation from an oscillating electric dipole:

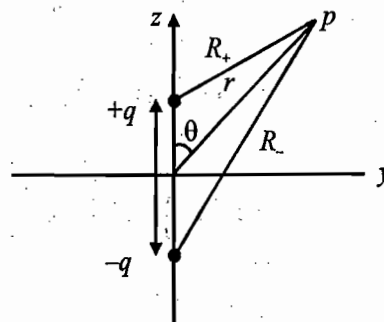
Suppose we consider two tiny metal spheres separated by a small distance ' ℓ ' and connected by a fine wire. The system is assumed to be as a whole electrically neutral. So, if at any time ' t ' the charge on the upper sphere is $q(t)$. We drive the charge back and forth through the wire, from one end to other end at an angular frequency ω .

$$q(t) = q_0 \cos \omega t$$

The oscillating dipole moment

$$p(t) = p_0 \cos(\omega t) \hat{z}$$

where, $p_0 = q_0 \ell$ is the maximum value of dipole moment.



Retarded Scalar Potential:

The retarded scalar potential at P at an instant t is given by

$$\phi(r, t) = \frac{1}{4\pi\epsilon_0} \left[\frac{q_0 \cos \omega(t - R_+/c)}{R_+} - \frac{q_0 \cos \omega(t - R_-/c)}{R_-} \right] \quad \dots (11)$$



For perfect dipole $\ell \ll r$ we can write

$$R_{\pm} = r \left(1 \mp \frac{1}{2} \frac{\ell}{r} \cos \theta \right)$$

Or,
$$\frac{1}{R_{\pm}} = \frac{1}{r} \left(1 \pm \frac{\ell}{2r} \cos \theta \right) \quad \dots (12)$$

We assume that $\ell \ll \lambda$ as $\ell \ll \frac{c}{\omega}$

$$\begin{aligned} \cos \omega \left(t - \frac{R_{\pm}}{c} \right) &= \cos \left[\omega \left(t - \frac{r}{c} \right) \pm \frac{\omega \ell}{2c} \cos \theta \right] \\ &= \cos \omega \left(t - \frac{r}{c} \right) \cos \left(\frac{\omega \ell}{2c} \cos \theta \right) \mp \sin \omega \left(t - \frac{r}{c} \right) \sin \left(\frac{\omega \ell}{2c} \cos \theta \right) \end{aligned}$$

$$\cos \omega \left(t - \frac{R_{\pm}}{c} \right) = \cos \omega \left(t - \frac{r}{c} \right) \mp \frac{\omega \ell}{2c} \cos \theta \sin \omega \left(t - \frac{r}{c} \right) \quad \dots (13)$$

Putting the equation (9) and (8) in equation (7), we get

$$\phi(r, t) = \frac{p_0 \cos \theta}{4\pi\epsilon_0 r} \left[-\frac{\omega}{c} \sin \left(t - \frac{r}{c} \right) + \frac{1}{r} \cos \omega \left(t - \frac{r}{c} \right) \right]$$

Let $r \gg \lambda$, $r \gg \left(\frac{c}{\omega} \right)$

$$\phi(r, t) = -\frac{p_0 \omega}{4\pi\epsilon_0 c} \left(\frac{\cos \theta}{r} \right) \sin \omega \left(t - \frac{r}{c} \right) \quad \dots (14)$$

Retardation vector potential:

The current, $I(t) = \frac{dq}{dt} = -q_0 \omega \sin \omega t$

Considering a current element, $\hat{k} I(t) dt$ at a distance z from the centre of the dipole. So, we can write for the retarded vector potential as

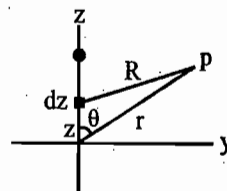
$$\vec{A}(r, t) = \hat{k} \frac{\mu_0}{4\pi} \int_{-\ell/2}^{\ell/2} \frac{-q_0 \omega \sin \omega t}{R} dz$$

$$R = \sqrt{r^2 + z^2 - 2zr \cos \theta}$$

Let $r \gg \ell$.

$$\therefore R = r \left(1 + \frac{z^2}{r^2} - \frac{2z}{r} \cos \theta \right)^{1/2} \approx r$$

$$\therefore \vec{A}(r, t) = -\hat{k} \frac{\mu_0}{4\pi} \frac{a_0 \omega}{r} \sin \omega \left(t - \frac{r}{c} \right) \int_{-\ell/2}^{\ell/2} dz$$





In spherical coordinate, $\hat{k} = (\cos \theta \hat{r} - \sin \theta \hat{\theta})$

$$\therefore \bar{A}(r, t) = -\frac{\mu_0 P_0 \omega}{4\pi r} \sin \omega \left(t - \frac{r}{c} \right) (\cos \theta \hat{r} - \sin \theta \hat{\theta}) \quad \dots (15)$$

$\vec{E}$ and $\vec{B}$ field due to dipole radiation :

$$\begin{aligned} \vec{\nabla} \phi &= \left(\frac{\partial \phi}{\partial r} \hat{r} + \frac{1}{r} \frac{\partial \phi}{\partial \theta} \hat{\theta} \right) \\ &= -\frac{P_0 \omega}{4\pi \epsilon_0 c} \left[\hat{r} \cos \theta \left\{ -\frac{1}{r^2} \sin \omega \left(t - \frac{r}{c} \right) - \frac{\omega}{cr} \cos \omega \left(t - \frac{r}{c} \right) \right\} - \frac{\sin \theta}{r^2} \sin \omega \left(t - \frac{r}{c} \right) \hat{\theta} \right] \end{aligned}$$

Since, $r \gg \frac{c}{\omega}$

$$\vec{\nabla} \phi = \hat{r} \frac{P_0 \omega^2}{4\pi r} \cos \theta \left(t - \frac{r}{c} \right) (\cos \theta \hat{r} - \sin \theta \hat{\theta}) \quad \dots (16)$$

And
$$\frac{\partial \bar{A}}{\partial t} = -\frac{\mu_0 P_0 \omega^2}{4\pi r} \cos \omega \left(t - \frac{r}{c} \right) (\cos \theta \hat{r} - \sin \theta \hat{\theta}) \quad \dots (17)$$

$$\therefore \vec{E} = -\vec{\nabla} \phi - \frac{\partial \bar{A}}{\partial t}$$

From equation (16) and (17), we get

Or,
$$\vec{E} = -\hat{\theta} \frac{\mu_0 P_0 \omega^2}{4\pi r} \left(\frac{\sin \theta}{r} \right) \cos \omega \left(t - \frac{r}{c} \right) \quad \dots (18)$$

$\vec{B}$ field:

$$\vec{\nabla} \times \bar{A} = \frac{1}{r^2 \sin \theta} \begin{vmatrix} \hat{r} & r\hat{\theta} & r\sin\theta\hat{\phi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ A_r & rA_\theta & r\sin\theta A_\phi \end{vmatrix}$$

$$= -\frac{\mu_0 P_0 \omega}{4\pi r^2 \sin \theta} \begin{vmatrix} \hat{r} & r\hat{\theta} & r\sin\theta\hat{\phi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ \sin \omega \left(t - \frac{r}{c} \right) & -\frac{\sin \omega \left(t - \frac{r}{c} \right)}{r} \sin \theta & 0 \end{vmatrix}$$

$$= -\frac{\mu_0 p_0 \omega}{4\pi r} \left[\frac{\omega}{c} \sin \theta \cos \omega \left(t - \frac{r}{c} \right) + \frac{\sin \theta}{r} \sin \omega \left(t - \frac{r}{c} \right) \right]$$

If $r \gg \frac{c}{\omega}$

$$\vec{B} = \vec{\nabla} \times \vec{A} = -\frac{\mu_0 p_0 \omega}{4\pi r c} \sin \theta \cos \omega \left(t - \frac{r}{c} \right) \hat{\phi}$$

$$\vec{B} = -\frac{\mu_0 p_0 \omega}{4\pi r c} \sin \theta \cos \omega \left(t - \frac{r}{c} \right) \hat{\phi} \quad \dots (19)$$

Radiation Power:

$$\vec{S} = (\vec{E} \times \vec{H}) = \frac{1}{\mu_0} (\vec{E} \times \vec{H})$$

$$= \hat{r} \frac{\mu_0}{c} \left[\frac{p_0 \omega^2}{4\pi} \left(\frac{\sin \theta}{r} \right) \cos \omega \left(t - \frac{r}{c} \right) \right]^2$$

Therefore, the intensity of radiation

$$I = \langle S \rangle = \frac{\mu_0 p_0^2 \omega^4}{32\pi^2 c} \frac{\sin^2 \theta}{r^2} \hat{r} \quad \dots (20)$$

So, there is no radiation along dipole axis and the intensity is maximum along the perpendicular to the dipole axis.

The total power radiated by the dipole is given by

$$P = \int \langle S \rangle dS = \frac{\mu_0 p_0^2 \omega^4}{32\pi^2 c} \int_0^{2\pi} \int_0^\pi \frac{\sin^2 \theta}{r^2} r^2 \sin \theta d\theta d\phi = \frac{\mu_0 p_0^2 \omega^4}{32\pi^2 c} \frac{4\pi}{3}$$

$$P = \frac{1}{4\pi\epsilon_0} \frac{p_0^2 \omega^4}{3c^3} \quad \dots (21)$$

10.3. Radiation from a point charge :

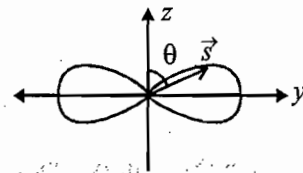
From Lienard Wiechert potential we can write the scalar and vector potential of a moving charge particle.

$$\phi(r, t) = \frac{1}{4\pi\epsilon_0} \frac{qc}{(rc - \vec{r} \cdot \vec{v})} \quad \dots (22)$$

$$\vec{A}(r, t) = \frac{\mu_0}{4\pi} \left(\frac{qc\vec{v}}{rc - \vec{r} \cdot \vec{v}} \right) = \frac{v}{c^2} \phi(r, t) \quad \dots (23)$$

$$\vec{E} = -\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t}$$

$$= -\frac{qc}{4\pi\epsilon_0} \frac{1}{(rc - \vec{r} \cdot \vec{v})} - \frac{qc}{4\pi\epsilon_0} \frac{1}{(rc - \vec{r} \cdot \vec{v})} \left[\frac{\vec{v}}{c} - \frac{\vec{r}}{c} \frac{d}{dt} \left(\frac{1}{rc - \vec{r} \cdot \vec{v}} \right) \right]$$





$$\vec{E}(r, t) = \frac{q}{4\pi\epsilon_0} \frac{r}{(\vec{r} \cdot \vec{u})^3} \left[(c^2 - v^2) \vec{u} + \vec{r} \times (\vec{u} \times \vec{a}) \right] \quad \dots (24)$$

Where, $u = (c\hat{r} - v)$

First term is called velocity field and second term is called acceleration field.

$$\begin{aligned} \vec{B}(r, t) &= (\vec{\nabla} \times \vec{A}) = \frac{1}{c^2} (\vec{\nabla} \times \phi \vec{v}) \\ &= \frac{1}{c^2} \left[\phi (\vec{\nabla} \times \vec{v}) - \vec{v} \times \vec{\nabla} \phi \right] \\ &= -\frac{1}{c^2} \frac{q}{4\pi\epsilon_0} \frac{1}{(\mu_0 r)^3} \vec{r} \times \left[(c^2 - u^2) \vec{v} + (\vec{r} - \vec{a}) \vec{v} + (\vec{r} - \vec{u}) \vec{a} \right] \\ &= \frac{1}{c} \hat{r} \times \vec{E}(r, t) \end{aligned} \quad \dots (25)$$

Therefore, the poynting vector is

$$\begin{aligned} \vec{S} &= \frac{1}{\mu_0} \vec{E} \times \vec{B} = \frac{1}{\mu_0 c} \left[\vec{E} \times (\hat{r} \times \vec{E}) \right] \\ \vec{S} &= \frac{1}{\mu_0 c} \left[E^2 \hat{r} - (\hat{r} \cdot \vec{E}) \vec{E} \right] \end{aligned} \quad \dots (26)$$

At only the acceleration fields represent the true rediation.

$$\therefore \vec{E}_{rad} = \frac{q}{4\pi\epsilon_0} \frac{r}{(\vec{r} \cdot \vec{u})^3} \left[\vec{r} \times (\vec{u} \times \vec{a}) \right] \quad \dots (27)$$

Since the radiation electric field perpendicular to the motion of point charge
Therefore, the equation (26) become

$$\vec{S} = \frac{1}{\mu_0 c} E_{rad}^2 \hat{r} \quad \dots (28)$$

If the charge is instantaneously at rest (at time t) then $\mu = c\hat{r}$.

$$\vec{E}_{rad} = \frac{q}{4\pi\epsilon_0 c^2 r} \left[\hat{r} \times (\hat{r} \times \vec{a}) \right] = \frac{\mu_0 q}{4\pi r} \left[(\hat{r} \cdot \vec{a}) \hat{r} - \vec{a} \right]$$

$$\begin{aligned} \vec{S}_{rad} &= \frac{1}{\mu_0 c} \left(\frac{\mu_0 q}{4\pi r} \right)^2 \left[a^2 - (\hat{r} \cdot \vec{a})^2 \right] \hat{r} \\ &= \frac{1}{\mu_0 c} \left(\frac{\mu_0 q}{4\pi r} \right)^2 a^2 \sin^2 \theta = \frac{\mu_0 q^2 a^2}{16\pi^2 c} \left(\frac{\sin^2 \theta}{r^2} \right) \hat{r} \end{aligned}$$

where θ is the angle between $\hat{r}$ and $\vec{a}$.

Therefore, no power is radiated in the forward or backward direction rather it is emitted in a donut the direction of instantaneous acceleration.

Therefore, total power radiated is



$$= \frac{\mu_0 q^2 a^2}{16\pi^2 c} \frac{4\pi}{3}$$

$$p = \frac{\mu_0 q^2 a^2}{6\pi c} \quad \dots (29)$$

Lorentz covariance of Maxwell's equations:

Let S and S' be two inertial frames, S' moving relative to S with a uniform velocity v along the common x (or x') direction. If a system of charges be fixed in S' -observer will observe an electric field only. But since the charges move relative to an S -observer, he will observe both an electric and a magnetic field. So, if an electromagnetic field is electric or magnetic depends on the inertial frame of observation. Thus, for invariance of Maxwell's field equations under Lorentz transformation, the electric and magnetic field components require relativistic transformation from one inertial frame to another.

Deduction of transformation formulae:

Maxwell's field equations in S -frame are

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0} \quad \dots (30)$$

$$\vec{\nabla} \cdot \vec{B} = 0 \quad \dots (31)$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad \dots (32)$$

And
$$\vec{\nabla} \times \vec{B} = \mu_0 \left(\vec{J} + \epsilon_0 \frac{\partial \vec{E}}{\partial t} \right) \quad \dots (33)$$

with usual meanings of the symbol.

In component forms (32) and (31) may be written as

$$\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = -\frac{\partial B_x}{\partial t} \quad \dots (34)$$

$$\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = -\frac{\partial B_y}{\partial t} \quad \dots (35)$$

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -\frac{\partial B_z}{\partial t} \quad \dots (36)$$

And
$$\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = 0 \quad \dots (37)$$

where E_x 's and B_x 's are respectively the electric and magnetic field components at (x, y, z, t) of S -frame. At the corresponding space-time coordinates (x', y', z', t') in S' -frame, the electric and magnetic fields $\vec{E}'$ and $\vec{B}'$ must satisfy, for invariance,

$$\vec{\nabla} \times \vec{E}' = -\frac{\partial \vec{B}'}{\partial t'} \quad \text{and} \quad \vec{\nabla} \cdot \vec{B}' = 0$$

In component forms, therefore



$$\frac{\partial E'_z}{\partial y'} - \frac{\partial E'_y}{\partial z'} = -\frac{\partial B'_x}{\partial t'} \quad \dots (38)$$

$$\frac{\partial E'_x}{\partial z'} - \frac{\partial E'_z}{\partial x'} = -\frac{\partial B'_y}{\partial t'} \quad \dots (39)$$

$$\frac{\partial E'_y}{\partial x'} - \frac{\partial E'_x}{\partial y'} = -\frac{\partial B'_z}{\partial t'} \quad \dots (40)$$

$$\text{And} \quad \frac{\partial B'_x}{\partial x'} + \frac{\partial B'_y}{\partial y'} + \frac{\partial B'_z}{\partial z'} = 0 \quad \dots (41)$$

But (x', y', z', t') of S' -frame are related to (x, y, z, t) of S -frame by Lorentz formulae as:

$$x' = \gamma(x - vt); \quad y' = y; \quad z' = z; \quad t' = \gamma\left(t - \frac{vx}{c^2}\right)$$

where, γ = Lorentz factor = $\frac{1}{\sqrt{1 - v^2/c^2}}$

$$\text{Now,} \quad \frac{\partial}{\partial x} = \frac{\partial x'}{\partial x} \frac{\partial}{\partial x'} + \frac{\partial y'}{\partial x} \frac{\partial}{\partial y'} + \frac{\partial z'}{\partial x} \frac{\partial}{\partial z'} + \frac{\partial t'}{\partial x} \frac{\partial}{\partial t'} \quad \dots (42)$$

and similarly, we may write for $\frac{\partial}{\partial y}$, $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial t}$

But, from Lorentz transformation formulae, we obtain

$$\frac{\partial x'}{\partial x} = \gamma; \quad \frac{\partial y'}{\partial x} = 0 = \frac{\partial z'}{\partial x}; \quad \frac{\partial t'}{\partial x} = -\gamma \frac{v}{c^2}$$

From equation (39), we get

$$\frac{\partial}{\partial x} = \gamma \left(\frac{\partial}{\partial x'} - \frac{v}{c^2} \frac{\partial}{\partial t'} \right) \quad \dots (43)$$

Similarly, it can be readily shown that,

$$\frac{\partial}{\partial y} = \frac{\partial}{\partial y'}; \quad \frac{\partial}{\partial z} = \frac{\partial}{\partial z'} \quad \dots (44)$$

$$\text{And} \quad \frac{\partial}{\partial t} = \gamma \left(\frac{\partial}{\partial t'} - v \frac{\partial}{\partial x'} \right) \quad \dots (45)$$

Substituting these results in (32)

$$\frac{\partial E_x}{\partial z'} - \frac{\partial}{\partial x'} \left[\gamma (E_z + vB_y) \right] = -\frac{\partial}{\partial t'} \left[\gamma \left(B_y + \frac{v}{c^2} E_z \right) \right] \quad \dots (46)$$

In order that (46) may go over to (39), we should put

$$E'_x = E_x, \quad E'_z = \gamma(E_z + vB_y) \quad \text{and} \quad B'_y = \gamma(B_y + vE_z/c^2)$$

Similarly, if we put $E'_x = E_x$, $E'_y = \gamma(E_y - vB_z)$ and $B'_z = \gamma(B_z + vE_y/c^2)$, we see that (36) goes over to (40). The relativistic transformation formulae of electromagnetic field components, except B_x , are thus obtained.



So, the complete set of transformation relations of field components from S to S' is

$$\begin{aligned} E'_x &= E_x & B'_x &= B_x \\ E'_y &= \gamma(E_y - vB_z) & B'_y &= \gamma(B_y + vE_z/c^2) \\ E'_z &= \gamma(E_z + vB_y) & B'_z &= \gamma(B_z - vE_y/c^2) \end{aligned} \quad \dots (40)$$

The conjugate set of transformation from S' to S is obtained by changing v to $-v$ and interchanging primes and unprimes. The relations are

$$\begin{aligned} E_x &= E'_x & B_x &= B'_x \\ E_y &= \gamma(E'_y + vB'_z) & B_y &= \gamma(B'_y - vE'_z/c^2) \\ E_z &= \gamma(E'_z - vB'_y) & B_z &= \gamma(B'_z + vE'_y/c^2) \end{aligned} \quad \dots (41)$$

(i) The relativistic transformations show that the component of both $\vec{E}$ and $\vec{B}$ parallel to the relative velocity of frames undergoes no change but the components transverse to the relative velocity change into a mixture of magnetic and electric fields.

(ii) In establishing the transformation formulae for $\vec{E}$ and $\vec{B}$, we used Maxwell's relations: $\vec{\nabla} \cdot \vec{B} = 0$ and

$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$ which do not contain ϵ_0 , μ_0 , ρ or $\vec{j}$. We could as well utilize the other two relations to arrive at the same result by invoking Lorentz transformation and correctly transforming ρ and $\vec{j}$, and assuming the invariance of electronic charge. But the process is relatively more involved.

Perpendicular of $\vec{E}$ and $\vec{B}$:

We have

$$\begin{aligned} \vec{E} \cdot \vec{B} &= E_x B_x + E_y B_y + E_z B_z \\ &= E'_x B'_x + \gamma^2 (E'_y + vB'_z)(B'_y - vE'_z/c^2) + \gamma^2 (E'_z - vB'_y)(B'_z + vE'_y/c^2) \\ &= E'_x B'_x + \gamma^2 [E'_y B'_y - vE'_y E'_z/c^2 + vB'_z B'_y - v^2 B'_z E'_z/c^2 + E'_z B'_z \\ &\quad + E'_z E'_y v/c^2 - vB'_y B'_z - v^2 B'_y E'_y/c^2] \\ &= E'_x B'_x + \gamma^2 (1 - v^2/c^2) (E'_y B'_y + E'_z B'_z) \\ &= E'_x B'_x + E'_y B'_y + E'_z B'_z \\ &= \vec{E}' \cdot \vec{B}' \end{aligned}$$

Therefore, $\vec{E}' \cdot \vec{B}'$ is invariant under Lorentz transformation. This implies that if $\vec{E}$ and $\vec{B}$ are perpendicular in one internal frame, they remain perpendicular in all inertial frames.

Show that $E^2 - c^2 B^2$ remain unchanged under Lorentz transformation:

$$\begin{aligned} E_x &= E'_x & B_x &= B'_x \\ E_y &= \gamma(E'_y + vB'_z) & B_y &= \gamma(B'_y - vE'_z/c^2) \end{aligned}$$



$$\begin{aligned}
 \text{Now,} & \quad E^2 - c^2 B^2 \\
 \Rightarrow & \quad E_x^2 + E_y^2 + E_z^2 - c^2 (B_x^2 + B_y^2 + B_z^2) \\
 \Rightarrow & \quad E_x'^2 - c^2 B_x'^2 + \gamma^2 (E_y'^2 + E_z'^2 - B_y'^2 c^2 - B_z'^2 c^2) \\
 \Rightarrow & \quad E_x'^2 - c^2 B_x'^2 + \gamma^2 [E_y'^2 + 2E_y' B_z' v + v^2 B_z'^2 + E_z'^2 - 2E_z' v B_y' + v^2 B_y'^2 \\
 & \quad - c^2 B_y'^2 + 2B_y' E_z' v - \frac{v^2 E_z'^2}{c^2} - B_z'^2 c^2 - 2B_z' E_y' v - \frac{v^2 E_y'^2}{c^2}] \\
 \Rightarrow & \quad E_x'^2 - c^2 B_x'^2 + \gamma^2 (1 - v^2/c^2) E_y'^2 + \gamma^2 (1 - v^2/c^2) E_z'^2 + \gamma^2 (v^2 - c^2) B_z'^2 + \gamma^2 (v^2 - c^2) B_y'^2 \\
 \Rightarrow & \quad E_x'^2 + E_y'^2 + E_z'^2 - c^2 (B_x'^2 + B_y'^2 + B_z'^2) \\
 \Rightarrow & \quad E'^2 - c^2 B'^2 \quad (\text{proved})
 \end{aligned}$$

Relative effect on charge density:

Suppose a metallic rod is in the S -frame and it has a linear charge density λ . When an observer will measure from the S' frame, he will see that charge density increase.

If to be the length of the rod the charge density $\lambda = \frac{q}{\ell_0}$

Due to relativistic effect the length of the rod in S' frame:

$$\ell'_0 = \ell_0 \sqrt{1 - v^2/c^2}$$

Therefore, charge density, $\lambda' = \frac{q}{\ell'_0} = \frac{q}{\ell_0 \sqrt{1 - v^2/c^2}}$

$$\boxed{\lambda' = \lambda \gamma}, \text{ where } \gamma = \frac{1}{\sqrt{1 - v^2/c^2}} \quad \dots (42)$$

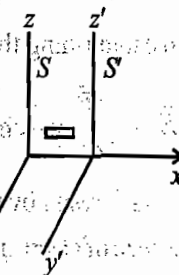
Relativistic Effect on current:

We current, $I_0 = \frac{\Delta q}{\Delta t_0}$ in S frame

Due to relativistic effect, $\Delta t' = \frac{\Delta t_0}{\sqrt{1 - v^2/c^2}}$

Therefore, current in S' frame, $I' = \frac{\Delta q}{\Delta t'} = \frac{\Delta q}{\Delta t / \sqrt{1 - v^2/c^2}}$

$$\boxed{I' = I_0 \sqrt{1 - v^2/c^2}} \quad \dots (43)$$





SOLVED PROBLEMS

1. For a vector potential $\vec{A}$, the divergence of $\vec{A}$ is $\vec{\nabla} \cdot \vec{A} = -\frac{\mu_0}{4\pi} \frac{Q}{r^2}$, where Q is a constant of appropriate dimension. The corresponding scalar potential $\phi(\vec{r}, t)$ that makes $\vec{A}$ and ϕ Lorentz gauge invariant is :

[GATE 2005]

- (a) $\frac{1}{4\pi\epsilon_0} \frac{Q}{r}$ (b) $\frac{1}{4\pi\epsilon_0} \frac{Qt}{r}$ (c) $\frac{1}{4\pi\epsilon_0} \frac{Q}{r^2}$ (d) $\frac{1}{4\pi\epsilon_0} \frac{Qt}{r^2}$

Soln. The Lorentz Gauge condition is

$$\vec{\nabla} \cdot \vec{A} + \mu_0 \epsilon_0 \frac{\partial \phi}{\partial t} = 0$$

$$\Rightarrow \mu_0 \epsilon_0 \frac{\partial \phi}{\partial t} = -\vec{\nabla} \cdot \vec{A}$$

$$\Rightarrow \mu_0 \epsilon_0 \frac{\partial \phi}{\partial t} = \frac{\mu_0}{4\pi} \frac{Q}{r^2}$$

$$\Rightarrow \int d\phi = \frac{1}{4\pi\epsilon_0} \int \frac{Q}{r^2} dt$$

$$\Rightarrow \phi = \frac{1}{4\pi\epsilon_0} \frac{Qt}{r^2}$$

Correct option is (d)

2. The electric (E) and magnetic (B) field amplitudes associated with an electromagnetic radiation from a point source behave at a distance r from the source as :

[GATE 2005]

- (a) $E = \text{constant}, B = \text{constant}$ (b) $E \propto \frac{1}{r}, B \propto \frac{1}{r}$
 (c) $E \propto \frac{1}{r^2}, B \propto \frac{1}{r^2}$ (d) $E \propto \frac{1}{r^3}, B \propto \frac{1}{r^3}$

Soln. We the field associated with a point source is

$$\vec{E}(r, t) = \frac{q}{4\pi\epsilon_0} \frac{\vec{r}}{(r \cdot u)^3} \left[(c^2 - v^2) \vec{u} + \vec{r} (\vec{u} \times \vec{a}) \right]$$

$$\Rightarrow \vec{E} \propto \frac{1}{r} \text{ and } \vec{B} = \frac{1}{c} \hat{r} \times \vec{E}(r, t)$$

$$\therefore \vec{B} \propto \frac{1}{r}$$

Correct option is (b)

3. The electric and the magnetic fields $\vec{E}(z, t)$ and $\vec{B}(z, t)$, respectively corresponding to the scalar potential $\phi(z, t) = 0$ and vector potential $\vec{A}(z, t) = \hat{i} z t$ are

[GATE 2012]

- (a) $\vec{E} = \hat{i} z$ and $\vec{B} = -\hat{j} t$ (b) $\vec{E} = \hat{i} z$ and $\vec{B} = \hat{j} t$ (c) $\vec{E} = -\hat{i} z$ and $\vec{B} = -\hat{j} t$ (d) $\vec{E} = -\hat{i} z$ and $\vec{B} = \hat{j} t$



$$\therefore \vec{E} = -\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t} = 0 - \hat{z}$$

$$\vec{E} = -\hat{z}$$

$$\vec{B} = \vec{\nabla} \times \vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ tz & 0 & 0 \end{vmatrix} = \hat{j}t$$

Correct option is (d)

4. Of A and V are vector and scalar potential $v(r, t) = 0$, $A(r, t) = \frac{-q}{4\pi\epsilon_0} \frac{t\vec{r}}{r^2}$. Scalar function $\lambda = \frac{-q}{4\pi\epsilon_0} \frac{t}{r^2}$

Due to gauge transformation transformed potential is

(a) $v'(r, t) = 0$ $A(r, t) = \frac{q}{4\pi\epsilon_0} \frac{t}{r^2}$ (b) $v'(r, t) = \frac{q}{4\pi\epsilon_0} \frac{t}{r^2}$ $A(r, t) = \frac{q}{4\pi\epsilon_0} \frac{t}{r^2}$
 (c) $v'(r, t) = \frac{-q}{4\pi\epsilon_0} \frac{1}{r^2}$ $A'(r, t) = \frac{-qt}{4\pi\epsilon_0} \frac{1}{r^2}$ (d) $v'(r, t) = \frac{q}{4\pi\epsilon_0} \frac{1}{r^2}$ $A'(r, t) = \frac{qt}{4\pi\epsilon_0} \frac{1}{r^3}$

Soln. We have,

$$v(r, t) = 0$$

$$\vec{A}(r, t) = \frac{q}{4\pi\epsilon_0} \frac{t\vec{r}}{r^3}$$

Scalar function, $\lambda = -\frac{q}{4\pi\epsilon_0} \frac{t}{r^2}$

The Gauge transformation is,

$$v'(r, t) = v(r, t) - \frac{\partial \lambda}{\partial t} = 0 + \frac{q}{4\pi\epsilon_0} \frac{1}{r^2}$$

And

$$\vec{A}'(r, t) = \vec{A}(r, t) - \vec{\nabla}\lambda = -\frac{q}{4\pi\epsilon_0} \frac{t}{r^3} \hat{r} + \frac{2q}{4\pi\epsilon_0} \frac{t}{r^3} \hat{r} = \frac{qt}{4\pi\epsilon_0 r^3} \hat{r}$$

Correct option is (d)

5. The angular distribution of the average power radiated from a short dipole antenna of length d , is dependent on d and polar angle θ . This dependence is proportional to

(a) $\sin^2 \theta$ (b) $d \sin^2 \theta$ (c) $d^2 \sin^2 \theta$ (d) $d \sin \theta$

Soln. Power radiated from the dipole antenna of length ' d ' is given by

$$P = \frac{\mu_0 p_0^2 v q \sin^2 \theta}{32\pi^2 c r^2} \propto d^2 \sin^2 \theta$$

Correct option is (c)



6. At time $t=0$, a charge distribution $\rho(\vec{r}, 0)$ exists within an ideal homogeneous conductor of permittivity ϵ and conductivity σ . At a later time $\rho(\vec{r}, t)$ is given by [GATE 2008]

$$\begin{aligned} \text{(a)} \quad \rho(\vec{r}, t) &= \rho(\vec{r}, 0) \exp\left(-\frac{\sigma t}{\epsilon}\right) & \text{(b)} \quad \rho(\vec{r}, t) &= \frac{\rho(\vec{r}, 0)}{1 + (\sigma t / \epsilon)^2} \\ \text{(c)} \quad \rho(\vec{r}, t) &= \rho(\vec{r}, 0) \exp\left[-\left(\frac{\sigma t}{\epsilon}\right)^2\right] & \text{(d)} \quad \rho(\vec{r}, t) &= \rho(\vec{r}, 0) \exp\left[\frac{\epsilon}{\sigma t} \sin\left(\frac{\sigma t}{\epsilon}\right)\right] \end{aligned}$$

Soln. We know that,

$$\vec{\nabla} \cdot \vec{J} + \frac{d\rho}{dt} = 0$$

$$\Rightarrow \vec{\nabla} \cdot \vec{J} = -\frac{d\rho}{dt}$$

$$\Rightarrow (\vec{\nabla} \cdot \vec{E}) = -\frac{d\rho}{dt} \quad (\because \vec{J} = \sigma \vec{E})$$

$$\Rightarrow \frac{\sigma \rho}{\epsilon} = -\frac{d\rho}{dt}$$

$$\Rightarrow \int \frac{d\rho}{\rho} = -\int \frac{\sigma}{\epsilon} dt$$

$$\Rightarrow \ln \rho = -\frac{\sigma}{\epsilon} t + c$$

$$\Rightarrow \rho(r, t) = \rho(r, 0) e^{-\frac{\sigma t}{\epsilon}}$$

$$\Rightarrow \rho(r, t) = \rho(r, 0) e^{-\frac{\sigma t}{\epsilon}}$$

Correct option is (a)



PRACTICE SET

1. Which of the following equations is relativistically invariant? (α, β, γ and δ are constants of suitable dimensions).

(a) $\frac{\partial \phi(x, t)}{\partial t} = \alpha \frac{\partial^2 \phi(x, t)}{\partial x^2}$

(b) $\frac{\partial^2 \phi(x, t)}{\partial t^2} = \beta \frac{\partial^2 \phi(x, t)}{\partial x^2}$

(c) $\frac{\partial^2 \phi(x, t)}{\partial t^2} = \gamma \frac{\partial \phi(x, t)}{\partial x}$

(d) $\frac{\partial^2 \phi(x, t)}{\partial t} = \delta \frac{\partial^3 \phi(x, t)}{\partial x^3}$

2. Let $\vec{A}$ and ϕ denote the vector and scalar potentials, respectively, which are Lorentz gauge invariant. $\vec{A}$ and ϕ satisfy.

(a) $\text{div} \vec{A} + \frac{1}{c} \frac{\partial \phi}{\partial t} = 0$

(b) $\text{div} \vec{A} + \frac{1}{c^2} \frac{\partial \phi}{\partial t} = 0$

(c) $\nabla^2 \vec{A} + \frac{1}{c} \frac{\partial \phi}{\partial t} = 0$

(d) $\nabla^2 \vec{A} + \frac{1}{c} \frac{\partial A}{\partial t} = 0$

3. The gauge transformations of the scalar and vector potentials that leave the electric and magnetic field invariant are

(a) $\vec{A} \rightarrow \vec{A}' = \vec{A} - \frac{\partial \Lambda}{\partial t}$

$\phi \rightarrow \phi' = \phi + \nabla \Lambda$

$\vec{A} \rightarrow \vec{A}' = \vec{A} - \nabla \Lambda$

(c) $\phi \rightarrow \phi' = \phi - \frac{\partial \Lambda}{\partial t}$

$\vec{A} \rightarrow \vec{A}' = \vec{A} + \nabla \Lambda$

(b) $\phi \rightarrow \phi' = \phi - \frac{\partial \Lambda}{\partial t}$

$\vec{A} \rightarrow \vec{A}' = \vec{A} + \nabla \Lambda$

(d) $\phi \rightarrow \phi' = \phi + \frac{\partial \Lambda}{\partial t}$

4. The angular distribution of the average power radiated from a short dipole antenna of length d , is dependent on d and polar angle θ . This dependence is proportional to

(a) $\sin^2 \theta$

(b) $d \sin^2 \theta$

(c) $d^2 \sin^2 \theta$

(d) $d \sin \theta$

5. Maxwell's equations can be written in terms of two coupled equations satisfied by the vector potential $\vec{A}$ and scalar potential ϕ . The Lorentz gauge which decouples these equations is given by

(a) $\vec{\nabla} \cdot \vec{A} = 0$

(b) $\nabla^2 \phi = \frac{1}{c^2} \frac{\partial^2 \vec{A}}{\partial t^2}$

(c) $\vec{\nabla} \cdot \vec{A} = -\frac{1}{c^2} \frac{\partial \phi}{\partial t}$

(d) $\vec{\nabla} \cdot \vec{A} = \frac{1}{c} \frac{\partial \phi}{\partial t}$

6. Of A and V are vector and scalar potential $v(r, t) = 0$, $\vec{A}(r, t) = \frac{-q}{4\pi\epsilon_0} \frac{t \vec{r}}{r^2}$. Scalar function $\lambda = \frac{-q}{4\pi\epsilon_0} \frac{t}{r^2}$

Due to gauge transformation transformed potential is

(a) $v'(r, t) = 0$ $A(r, t) = \frac{q}{4\pi\epsilon_0} t \frac{1}{r^2}$

(b) $v'(r, t) = \frac{q}{4\pi\epsilon_0} \frac{1}{r^2}$ $A(r, t) = \frac{q}{4\pi\epsilon_0} t \frac{1}{r^2}$

(c) $v'(r, t) = \frac{-q}{4\pi\epsilon_0} \frac{1}{r^2}$ $A'(r, t) = \frac{-qt}{4\pi\epsilon_0} \frac{1}{r^2}$

(d) $v'(r, t) = \frac{q}{4\pi\epsilon_0} \frac{1}{r^2}$ $A'(r, t) = \frac{qt}{4\pi\epsilon_0} \frac{1}{r^3}$

ANSWER KEY

Questions	1	2	3	4	5	6
Option	(b)	(b)	(b)	(c)	(c)	(d)

Chapter 15

Circuit Theory

Basic circuit

The emf of a battery is equal to the voltage across its terminals when the current is zero. That is, the emf is equivalent to the **open-circuit voltage** of the battery. The **equivalent resistance** of a set of resistors connected in series is

$$R_{eq} = R_1 + R_2 + R_3 + \dots \quad \dots (1)$$

The **equivalent resistance** of a set of resistors connected in parallel is

$$\frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \dots \quad \dots (2)$$

If it is possible to combine resistors into series or parallel equivalents, the preceding two equations make it easy to determine how the resistors influence the rest of the circuit.

Circuits involving more than one loop are conveniently analyzed with the use **Kirchhoff's rules**.

(1) The sum of currents entering any junction in an electric circuit must equal the sum of the currents leaving that junction:

$$\sum I_{in} = \sum I_{out} \quad \dots (3)$$

(2) The sum of the potential differences across all elements around any circuit loop must be zero.

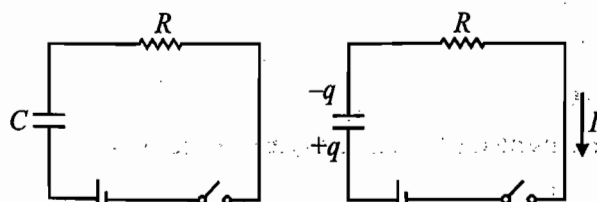
$$\sum_{\text{closed loop}} \Delta V = 0 \quad \dots (4)$$

The first rule is a statement of conservation of charge; the second is equivalent to a statement of conservation of energy.

When a resistor is traversed in the direction of the current, the change in potential ΔV across the resistor is $-IR$. When a resistor is traversed in the direction opposite the current, $\Delta V = +IR$. When a source of emf is traversed in the direction of the emf (negative terminal to positive terminal), the change in potential is $+\mathcal{E}$. When a source of emf is traversed opposite the emf (positive to negative), the change in potential is $-\mathcal{E}$. The use of these rules allows you to analyze electric circuits.

RC Circuit

Figure represents a simple RC circuit with DC voltage. If $\mathcal{E}$ be the voltage applied, and C and R are the capacitance and Resistance then the circuit equation is





$$\varepsilon - \frac{q}{C} - IR = 0$$

Substituting, $I = dq/dt$ we get in above equation

$$\frac{dq}{dt} = \frac{\varepsilon}{R} - \frac{q}{RC}$$

To find an expression for q , we first combine the terms on the right-hand side:

$$\frac{dq}{dt} = \frac{C\varepsilon}{RC} - \frac{q}{RC} = -\frac{q - C\varepsilon}{RC}$$

Now we multiply by dt and divide by $q - C\varepsilon$ to obtain

$$\frac{dq}{q - C\varepsilon} = -\frac{1}{RC} dt$$

Integrating this expression, using the fact that $q = 0$ at $t = 0$, we obtain

$$\int_0^q \frac{dq}{q - C\varepsilon} = -\frac{1}{RC} \int_0^t dt$$

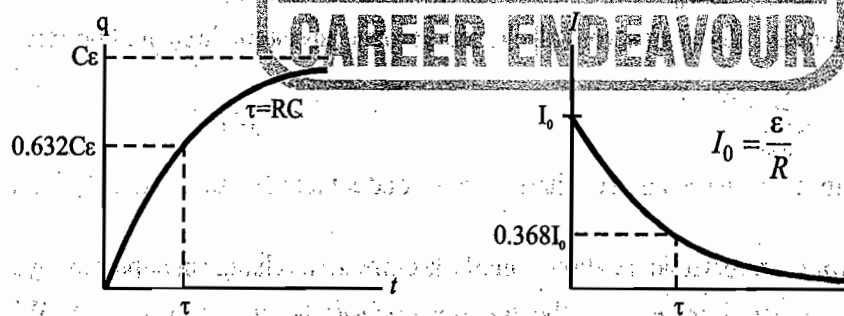
$$\ln \left(\frac{q - C\varepsilon}{-C\varepsilon} \right) = -\frac{1}{RC} t$$

Or, $q(t) = C\varepsilon (1 - e^{-t/RC}) = Q(1 - e^{-t/RC})$ (5)

By differentiating the expression we get the current

$$I(t) = \frac{\varepsilon}{R} e^{-t/RC}$$
 (6)

Now we can represent q and I graphically



While discharging, no voltage in the circuit. So, the circuit equation becomes

$$-\frac{q}{C} - IR = 0$$

When we substitute $I = dq/dt$ into this expression, it becomes

$$-R \frac{dq}{dt} = \frac{q}{C}$$

$$\frac{dq}{q} = -\frac{1}{RC} dt$$

Integrating this expression, using the fact that $q = Q$ at $t = 0$, gives



$$\ln\left(\frac{q}{Q}\right) = -\frac{t}{RC}$$

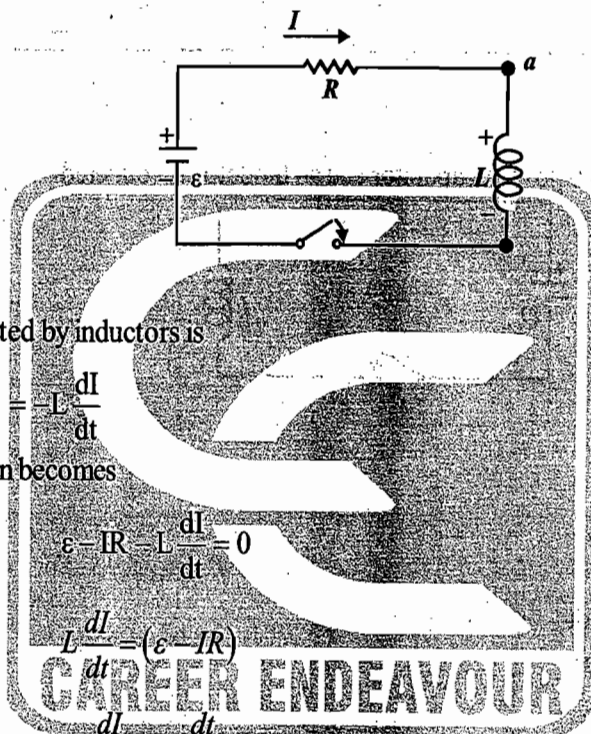
$$q(t) = Qe^{-t/RC} \quad \dots (7)$$

Again differentiating we get the expression of I .

$$I(t) = \frac{dq}{dt} = \frac{d}{dt}(Qe^{-t/RC}) = -\frac{Q}{RC}e^{-t/RC} \quad \dots (8)$$

LR Circuit

Consider an inductor of inductance L and a resistor with resistance R connected to a battery



The back emf created by inductors is

$$\varepsilon_L = -L \frac{dI}{dt}$$

The circuit equation becomes

$$\varepsilon - IR - L \frac{dI}{dt} = 0$$

$$\Rightarrow L \frac{dI}{dt} = (\varepsilon - IR)$$

$$\Rightarrow \frac{dI}{(\varepsilon - IR)} = \frac{dt}{L}$$

Integrating this equation, we get

$$\ln\left(\frac{\varepsilon - IR}{\varepsilon}\right) = -\frac{R}{L}t$$

$$\Rightarrow \frac{\varepsilon - IR}{\varepsilon} = e^{-Rt/L} \Rightarrow I = \frac{\varepsilon}{R}(1 - e^{-Rt/L})$$

$$\Rightarrow I = I_0(1 - e^{-Rt/L}) \quad \dots (9)$$

$$\text{where, } I_0 = \frac{\varepsilon}{R}$$

We can also write this expression as

$$I = \frac{\varepsilon}{R}(1 - e^{-t/\tau}) \quad \dots (10)$$

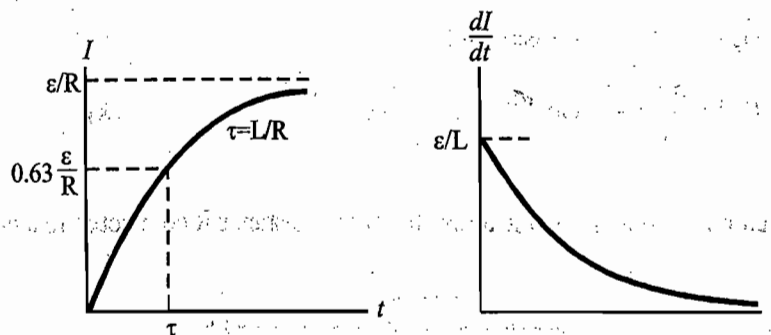
Where the constant τ is the time constant of the RL circuit:

$$\tau = L/R$$

Physically, τ is the time it takes the current in the circuit to reach $(1 - e^{-1}) = 0.63$ of its final value ε/R . The time constant is a useful parameter for

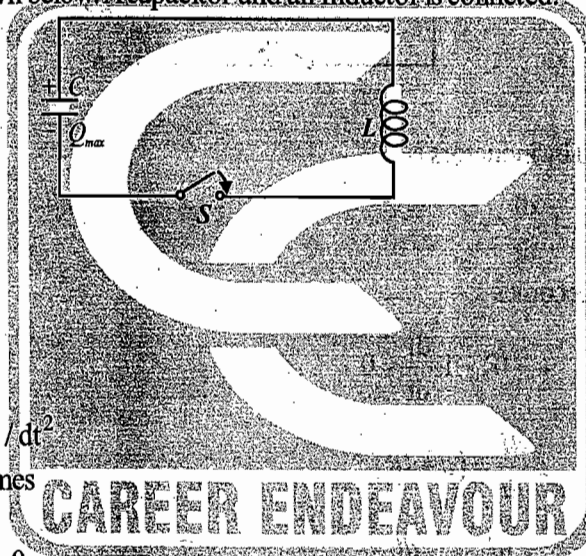
$$\frac{dI}{dt} = \frac{\varepsilon}{L} e^{-t/\tau} \quad \dots (12)$$

Graphically



LC circuit

Consider a LC circuit as shown below. A capacitor and an Inductor is conneted.



The circuit equation is

$$\frac{Q}{C} + L \frac{dI}{dt} = 0$$

We know

$$dI/dt = d^2Q/dt^2$$

So, the circuit equation becomes

$$\frac{Q}{C} + L \frac{d^2Q}{dt^2} = 0$$

$$\frac{d^2Q}{dt^2} = -\frac{1}{LC} Q$$

This is the same equation of simple harmonic oscillator.

$$\frac{d^2Q}{dt^2} = -\frac{k}{m} x = -\omega^2 x$$

So, solving for Q we get

$$Q = Q_{\max} \cos(\omega t + \phi) \quad \dots (13)$$

where $Q_{\max}$ is the maximum charge of the capacitor and the angular frequency ω is

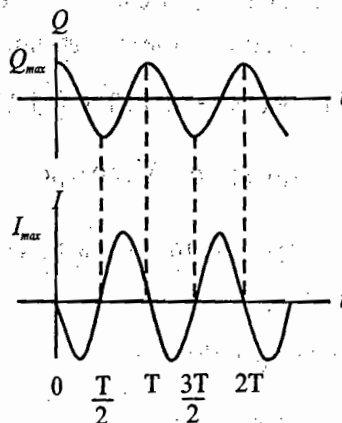
$$\omega = \frac{1}{\sqrt{LC}} \quad \dots (14)$$

Note that the angular frequency of the oscillations depends slowly on the inductance and capacitance of the circuit. This is the natural frequency of oscillation of the LC circuit.

Differentiating Q, we get I

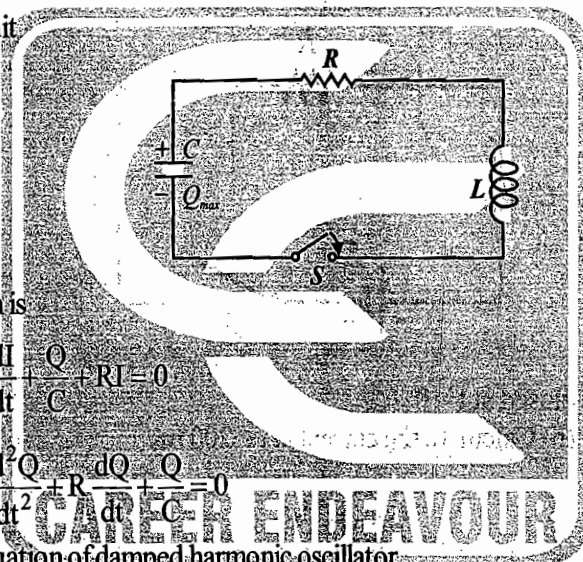
$$= \frac{dQ}{dt} = -\omega Q_{\max} \sin(\omega t + \phi) \quad \dots (15)$$

Charge and current varies like



RLC circuit

Shows a RLC circuit



The circuit equation is

$$L \frac{dI}{dt} + \frac{Q}{C} + RI = 0$$

Or,

$$L \frac{d^2Q}{dt^2} + R \frac{dQ}{dt} + \frac{Q}{C} = 0$$

This is the same equation of damped harmonic oscillator.

$$m \frac{d^2x}{dt^2} + b \frac{dx}{dt} + kx = 0$$

So solving for small R we get

$$Q = Q_{\max} e^{-Rt/2L} \cos \omega_d t \quad \dots (16)$$

Where,

$$\omega_d = \left[\frac{1}{LC} - \left(\frac{R}{2L} \right)^2 \right]^{1/2} \quad \dots (17)$$



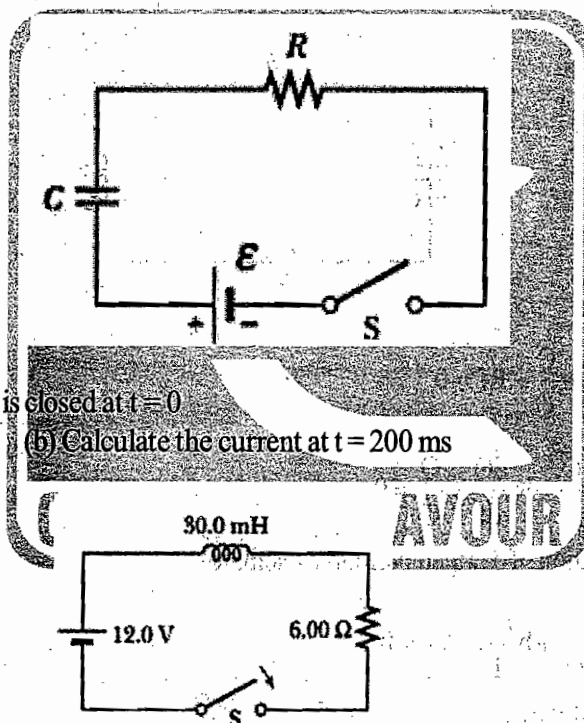
SOLVED PROBLEMS

1. An unchanged capacitor and a resistor are connected in series to battery, as shown in figure 8.8. If $\varepsilon = 12.0\text{ V}$, $C = 5.00\text{ }\mu\text{F}$ and $R = 8.00 \times 10^5\text{ }\Omega$, find the time constant of the circuit, the maximum charge on the capacitor, the maximum current in the circuit, and the charge and net current as functions of time.

Soln. The time constant of the circuit is $\tau = RC = (8.00 \times 10^5\text{ }\Omega)(5.00 \times 10^{-6}\text{ F}) = 4.00\text{ s}$. The maximum charge on the capacitor is $Q = C\varepsilon = (5.00\text{ }\mu\text{F})(12.0\text{ V}) = 60.0\text{ }\mu\text{C}$. The maximum current in the circuit is $I_0 = \varepsilon/R = (12.0\text{ V})/(8.00 \times 10^5\text{ }\Omega) = 15.0\text{ }\mu\text{A}$. Using these values and equation and, we find that

$$q(t) = (60.0\text{ }\mu\text{C})(1 - e^{-t/4.00\text{ s}})$$

$$I(t) = (15.0\text{ }\mu\text{A})e^{-t/4.00\text{ s}}$$

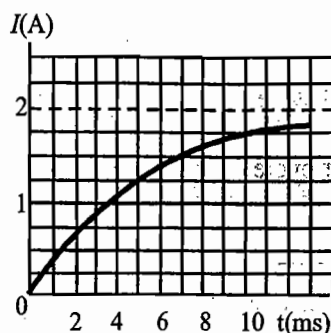


2. The time constant of Fig. 8.9 is closed at $t = 0$
 (a) Find the time constant (b) Calculate the current at $t = 200\text{ ms}$

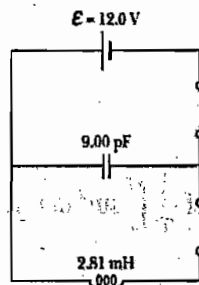
Soln. (a) The time constant is

$$\tau = \frac{L}{R} = \frac{30.0 \times 10^{-3}\text{ H}}{6.00\text{ }\Omega} = 5.00\text{ ms}$$

(b) The current is, $I = \frac{\varepsilon}{R}(1 - e^{-t/\tau}) = \frac{12.0\text{ V}}{6.00\text{ }\Omega}(1 - e^{-0.400}) = 0.659\text{ A}$



2. (a) Find the frequency of oscillation of the circuit.
 (b) What are the maximum values of charge on the capacitor and current in the circuit?
 (c) Determine the charge and current as functions of time.



Soln. (a) $f = \frac{\omega}{2\pi} = \frac{1}{2\pi\sqrt{LC}} = \frac{1}{2\pi(2.81 \times 10^{-3} \text{ H})(9.00 \times 10^{-12} \text{ F})^{1/2}} = 1.00 \times 10^6 \text{ Hz}$

(b) $Q_{\max} = C\varepsilon = (9.00 \times 10^{-12} \text{ F})(12.0 \text{ V}) = 1.08 \times 10^{-10} \text{ C}$

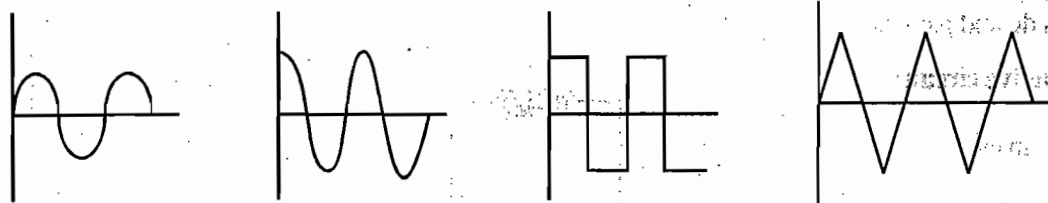
$I_{\max} = \omega Q_{\max} = 2\pi f Q_{\max} = (2\pi \times 10^6 \text{ s}^{-1})(1.08 \times 10^{-10} \text{ C}) = 6.49 \times 10^{-4} \text{ A}$

(c) $Q = Q_{\max} \cos \omega t = (1.08 \times 10^{-10} \text{ C}) \cos [2\pi \times 10^6 \text{ rad/s} \cdot t]$

$I = -I_{\max} \sin \omega t = (-6.49 \times 10^{-4} \text{ A}) \sin [2\pi \times 10^6 \text{ rad/s} \cdot t]$

AC circuit(Alternating current):

If the direction of the current in a resistor or in any other element changes alternatively, the current is called an alternative current (AC).



- Harmonic current is a current on which current is either sine or cosine,

House Hold supply:

$$V = V_0 \sin \omega t$$

$$\omega = 2\pi f; \quad f = 50 \text{ Hz}$$



$$V_{\text{avg}} = \frac{1}{T} \int_0^T V_0 \sin \omega t dt$$

$$= 0 \quad \text{for a full cycle.}$$

$$V_{\text{avg}} = \frac{2V_0}{\pi} \quad \text{for half cycle.}$$

R.M.S value:

$$V_{\text{rms}} = \sqrt{\frac{1}{T} \int_0^T (V_0 \sin \omega t)^2 dt} = V_0 \sqrt{\frac{1}{2T} \times T}$$

$$= \frac{V_0}{\sqrt{2}} = \frac{311}{\sqrt{2}} \approx 220 \text{ V}$$

$$I_{\text{rms}} = I_0 / \sqrt{2}$$

Pure resistive circuit:

$$iR = V_0 \sin \omega t; i = i_0 \sin \omega t$$

$\Rightarrow$ Current and voltage are in same phase.

$$H = \int i^2 R dt = R \int i_0^2 \sin^2 \omega t dt = i_0^2 RT \frac{1}{T} \int_0^T \sin^2 \omega t dt = \frac{i_0^2 RT}{2} = \left(\frac{i_0}{\sqrt{2}} \right)^2 RT$$

$$H = i_{\text{rms}}^2 RT$$

2. Pure capacitive circuit:

$$Q = CV$$

$$Q = CV_0 \sin \omega t$$

$$\frac{dQ}{dt} = \omega CV_0 \cos \omega t$$

$$i = \frac{V_0}{\omega C} \sin(\omega t + \pi/2)$$

$$i = \frac{V_0}{|X_C|} \sin(\omega t + \pi/2)$$

Where $|X_C| = \frac{1}{\omega C}$ reactive (capacitive)

$$X_C = \frac{1}{j\omega C}$$

Here current leads voltage by $\pi/2$.

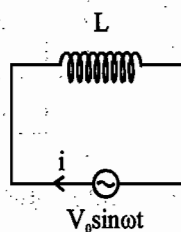
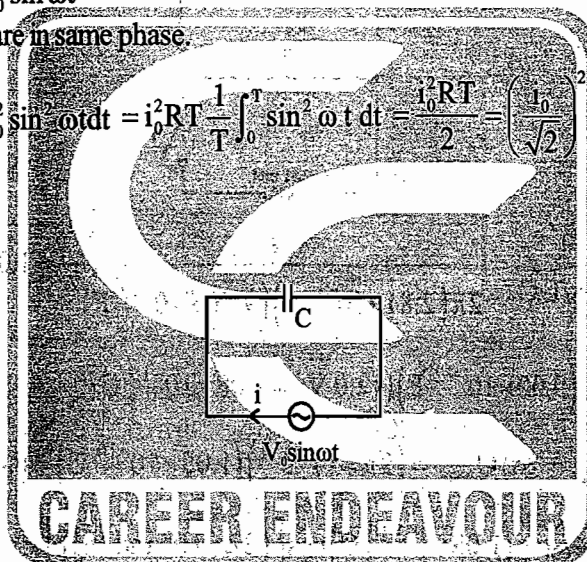
- It is a high pass filter.
- It blocks dc and pass a.c

3. Pure inductive circuit:

$$L \frac{di}{dt} = V_0 \sin \omega t$$

$$di = \frac{V_0}{L} \sin \omega t dt$$

$$i = \frac{V_0}{\omega L} \sin(\omega t - \pi/2)$$





Where $|X_L| = \omega L \rightarrow$ reactance (inductive) $X_L = j\omega L$

$\Rightarrow$ current in lags behind the voltage by $\frac{\pi}{2}$

• Low pass filter.

• Easy path for d.c. and hard for a.c.

4. **Series:**

L-R circuit:

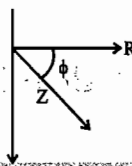
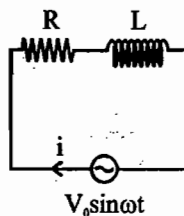
$$V = V_0 \sin \omega t$$

$$i = \frac{V_0}{Z} \sin(\omega t + \phi)$$

where $z \rightarrow$ impedance

$$z = \sqrt{R^2 + X_L^2} = \sqrt{R^2 + \omega^2 L^2}$$

$$\phi = \tan^{-1} \left(\frac{\omega L}{R} \right)$$



5. **Series C-R circuit:**

$$V = V_0 \sin \omega t$$

$$i = \frac{V_0}{Z} \sin(\omega t + \phi)$$

$$Z = \sqrt{R^2 + X_C^2}$$

$$\tan \phi = \frac{X_C}{R}$$

6. **Series LC circuit:**

$$V = V_0 \sin \omega t$$

$$i = \frac{V_0}{Z} \sin(\omega t + \phi)$$

$$Z = |X_C + X_L| = \left| \left(\frac{1}{\omega C} - \omega L \right) \right|$$

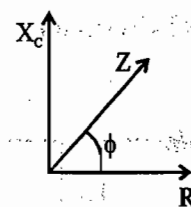
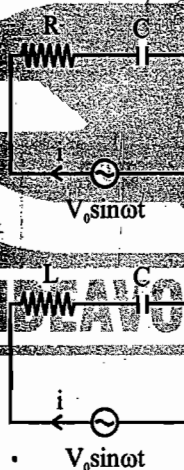
$\phi = +\frac{\pi}{2}$ if $X_C > X_L$ (Capacitance circuit)

$\phi = -\frac{\pi}{2}$ if $X_L > X_C$ (Inductive circuit)



If $X_L = X_C$
 $Z = 0 \Rightarrow i = \infty$ } [Resonance condition (Pure resistive circuit)]

If $X_L = X_C$ & $V = 0$; $i = \frac{V}{Z} = \frac{0}{0}$ indeterminant



7. Series LCR circuit:

$$Z = \sqrt{(X_L - X_C)^2 + R^2}; \tan \phi = \frac{X_L - X_C}{R}$$

$$X_C = \frac{1}{\omega C}; X_L = \omega L$$

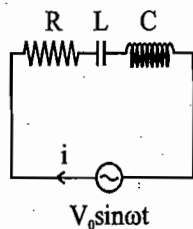
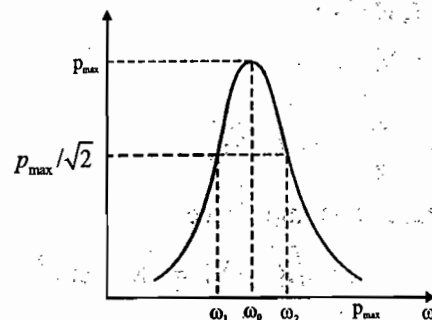
$$\text{And, } V = \sqrt{V_R^2 + (V_C - V_L)^2}$$

At resonance frequency

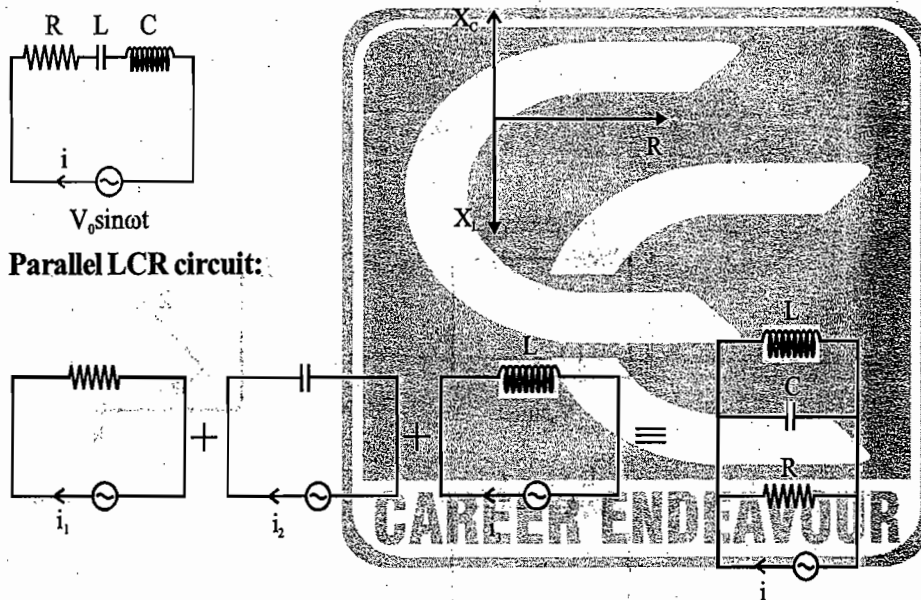
$$|X_L| = |X_C|; \omega_0 L = \frac{1}{\omega_0 C}; \omega_0 = \frac{1}{\sqrt{LC}}$$

$$\Delta \omega = \frac{\omega_0}{Q}$$

$$Q = \text{Quality factor} = \frac{\omega_0 L}{R} = \frac{1}{\omega_0 C R} \Rightarrow |Q V_R| = |V_L| = |V_C|$$



Parallel LCR circuit:



$$i_1 = \frac{V_0}{R} \sin \omega t$$

$$i_2 = \frac{-V_0}{X_L} \cos \omega t, \quad i_3 = \frac{V_0}{X_C} \cos \omega t$$

$$i = i_1 + i_2 + i_3$$

Power consumption:

$$P = \frac{1}{T} \int_0^T V i dt = \frac{1}{T} \int_0^T V_0 \sin \omega t i_0 \sin(\omega t + \phi) dt$$

$$P = V_{\text{rms}} I_{\text{rms}} \cos \phi$$

Where $\cos \phi \rightarrow$ phase factor equal to $\frac{R}{|Z|}$

- For pure resistive circuit $\phi = 0$
- For pure inductive, $\phi = -\pi/2; P = 0$



• In tubelight, we prefer choke coil over a resistance to control the current because power consumption by induction is minimum.

4. A $150\ \Omega$ resistor, a $10\ \mu\text{F}$ capacitor and a $0.1\ \text{H}$ inductor are connected in series to an A.C. source operating at an angular frequency ω .
- (a) Find the value of ω for which the combination acts as a pure resistive load.
- (b) The a.c. source is operated at a peak voltage of $300\sqrt{2}$ and a frequency equal to half of the resonance frequency of the circuit. Find the peak value of the current in the ckt and the phase difference between the current and voltage. Also find the peak voltage across the inductor.

Soln. At resonant state $X_L = X_C$ (Pure resistive circuit)

Given that $R = 150\ \Omega$, $C = 10\ \mu\text{F} = 10 \times 10^{-6}\ \text{F}$

$$L = 0.1\ \text{H}; X_L = X_C; \omega L = \frac{1}{\omega C}; \omega^2 = \frac{1}{LC} = \frac{1}{0.1 \times 10^{-5}} = 10^6; \omega = 1000\ \text{rad/sec}$$

(b) $V_0 = 300\sqrt{2}$

Now, $\omega' = \frac{\omega}{2} = 500\ \text{rad/sec}$

$$Z = \sqrt{(X_L - X_C)^2 + R^2}$$

$$X_C = \omega' L = 500 \times 0.1 = 50$$

$$X_L = \frac{1}{\omega' C} = \frac{1}{500 \times 10^{-5}} = 200$$

$$\Rightarrow Z = \sqrt{(200 - 50)^2 + (150)^2} = 150\sqrt{2}$$

Peak value of current

$$i_0 = \frac{V_0}{Z} = \frac{300\sqrt{2}}{150\sqrt{2}} = 2\ \text{A}$$

$$\text{Phase difference, } \tan \phi = \frac{X_C - X_L}{R} = \frac{200 - 50}{150} = 1$$

$$\Rightarrow \phi = 45^\circ$$

$$\Rightarrow \text{current leads by } 45^\circ.$$

Peak voltage across the inductor.

$$X_L = \omega L = 50; V_0 = i_0 X_L = 2 \times 50 = 100\ \text{volt}$$

5. A resistor of $1\ \text{k}\Omega$ and an inductor of $5\ \mu\text{H}$ are connected in series with a battery of emf. $4\ \text{V}$ through a switch. The switch is closed at $t = 0$. In the following, you may use $e^3 = 20$
- (a) Find the current flowing in the circuit at $t = 15\ \mu\text{sec}$.
- (b) Find the heat dissipated through the resistor during the first $15\ \mu\text{s}$.

[JAM - 2006]

Soln. Given that $R = 1\ \text{k}\Omega = 10^3\ \Omega$

$$L = 5\ \text{mH} = 5 \times 10^{-3}\ \text{H}$$

$$\text{emf} = 4\ \text{V}$$

$$t = 15\ \mu\text{sec} = 15 \times 10^{-6}$$

(a) Growth of current i through the circuit is given by

$$i = i_0 (1 - e^{-\lambda t})$$



$$\text{where } i_0 = \frac{\text{emf}}{R} = \frac{V}{R}$$

$$\text{and } \lambda = \frac{L}{R} = \text{time constant of the circuit}$$

$$\text{Here, } i_0 = \frac{4}{10^3} = 4 \times 10^{-3} \text{ A}; \lambda = \frac{L}{R} = \frac{5 \times 10^{-3}}{10^3} = 5 \times 10^{-6}$$

$$\Rightarrow i = 4 \times 10^{-3} \left(1 - e^{-\frac{t}{\lambda}} \right) = 4 \times 10^{-3} \left(1 - \frac{1}{20} \right) = \frac{4 \times 19}{20} \times 10^{-3} = 3.8 \times 10^{-3} \text{ amp.}$$

(b) Heat dissipated to the resistor in time t

$$H = i^2 R t = (3.8 \times 10^{-3})^2 \times 1 \times 10^3 \times 15 \times 10^{-6} = 216.6 \times 10^{-9} = 2.166 \times 10^{-7} \text{ J}$$

6. Consider a single loop circuit with a resistor ($R = 160 \Omega$), a capacitor ($C = 10 \mu\text{F}$), an inductor ($L = 200 \text{ mH}$) and an alternating source with $E_{\text{max}} = 36 \text{ V}$. If the circuit is driven at frequency $f = 50 \text{ Hz}$.

(a) Find the impedance z for the circuit.

(b) Find the amplitude of the current I and phase constant ϕ

[JNU - 2006]

Soln. Given $R = 160 \Omega$, $C = 10 \mu\text{F} = 10^{-5} \text{ F}$

$$L = 200 \text{ mH} = 0.2 \text{ H}$$

$$f = 50 \text{ Hz}, E_{\text{max}} = 36 \text{ V}$$

$$X_L = \omega L = 2\pi f L = 2 \times 3.14 \times 50 \times 0.2 = 62.8 \Omega$$

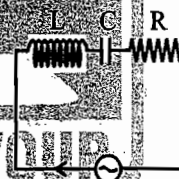
$$X_C = \frac{1}{\omega C} = \frac{1}{2\pi f C} = \frac{1}{2 \times 3.14 \times 50 \times 10^{-5}} = 318.47 \Omega$$

$$\begin{aligned} \text{(a) Impedance } z &= \sqrt{(X_C - X_L)^2 + R^2} \\ &= \sqrt{(318.47 - 62.8)^2 + 160^2} \\ &= \sqrt{65367.1489 + 25600} = \sqrt{90967.15} \Omega \end{aligned}$$

(b) Current in circuit (I_{rms}).

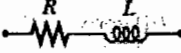
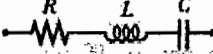
$$I_0 = \frac{E_{\text{max}}}{Z} = 36; I_{\text{rms}} = \frac{I_0}{\sqrt{2}}$$

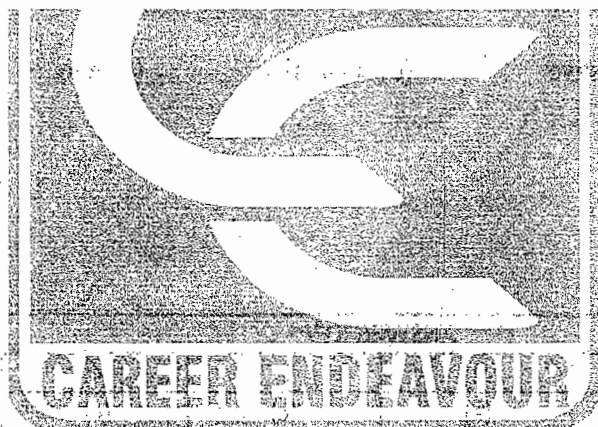
$$\text{Phase, } \tan \phi = \frac{X_C - X_L}{R} = \frac{318.47 - 62.8}{160}$$





SUMMARY TABLE

TABLE Impedance Values and Phase Angles for Various Circuit-Element Combinations ^a		
Circuit Elements	Impedance Z	Phase Angle ϕ
	R	0°
	X_C	-90°
	X_L	$+90^\circ$
	$\sqrt{R^2 + X_C^2}$	Negative, between -90° and 0°
	$\sqrt{R^2 + X_L^2}$	Positive, between 0° and 90°
	$\sqrt{R^2 + (X_L - X_C)^2}$	Negative if $X_C > X_L$ Positive if $X_C < X_L$





PRACTICE SET

- In an ac circuit the current and the emf are $i = 5 \sin(100\pi t - \pi/2) A$ and $v = 200 \sin(100\pi t) V$. Then the power dissipation in the circuit is
 (a) 1000W (b) 40W (c) 500W (d) zero
- In a pure inductive circuit impressed with an ac emf ε current
 (a) lags behind ε by 90° (b) is in phase with ε
 (c) leads over ε by 90° (d) is 180° out of phase with ε
- A capacitor acts as an open circuit for
 (a) ac (b) both ac and dc (c) dc (d) none of these
- The average value of the ac voltage $v = V_0 \sin \omega t$ over a complete period is
 (a) $\frac{2V_0}{\pi}$ (b) $\frac{V_0}{2}$ (c) $\frac{V_0}{\pi}$ (d) zero
- A pure LC parallel circuit is impressed with an ac source of frequency ω . If $\omega > \frac{1}{\sqrt{LC}}$, then the circuit reactance is
 (a) inductive (b) resistive (c) capacitive (d) zero

ANSWER KEY

Questions	1	2	3	4	5
Option	(d)	(a)	(c)	(d)	(c)

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